Analysis on infinite configuration spaces, and L^2-Betti numbers associated with infinite particle systems

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Based on joint works with
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Introduction: What is a Configuration Space?

X - topological space (e.g. $\mathbb{R}^d$ or Riemannian manifold)
Γ_X - space of locally finite subsets (configurations) in X :

$$\Gamma_X = \{\gamma \subset X : \ |\gamma \cap \Lambda| < \infty, \ \Lambda \ \mathit{compact}\}$$

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- Statistical mechanics (models of gases) - Dobrushin 56, 70; Ruelle 63, 70)
- Quantum Field Theory - Goldin, Grodnik, Powers, Sharp 75
- Representation Theory - Gelfand, Graev, Vershik 75
- Probability (theory of point processes) - Föllmer 75; Preston 76, 79; Georgii 76
- Topology - Fadell 62; Bödingheimer (spaces of *finite configurations*)

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Γ_X as *infinite dimensional manifold* - Albeverio, Kondratiev, Röckner 96

Topology and measures

$$\Gamma_X \ni \gamma \simeq \sum_{x\in\gamma} \delta_x \in \mathcal{M}(X), \quad \delta_x\text{- Dirac measure}$$

Vague topology on Γ_X - weak topology induced from $\mathcal{M}(X)$

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Poisson Measure π_σ (with reference measure σ)

1. Consider space of finite configurations in a compact $\Lambda \subset X$

$$\Gamma_\Lambda = \cup_n \widetilde{\Lambda^n}/S_n;$$

   define *probability measure* π_σ^Λ on Γ_Λ via formula

$$\pi_\sigma^\Lambda = e^{-\sigma(\Lambda)} \sum_n \frac{1}{n!}\sigma^n \quad (\sigma\text{- infinite measure on } X)$$

2. Define

$$\pi_\sigma = lim\ \pi_\sigma^\Lambda,\ \Lambda \uparrow X.$$

Quasi-invariance and gradient

- $Diff_0(X)$- group of compactly supported diffeomorphisms of X- acts on Γ_X:

$$\{..., x, y, z, ...\} \mapsto \{..., \varphi(x), \varphi(y), \varphi(z), ...\}, \quad \varphi \in Diff_0(X);$$

  Main fact: π_σ is $Diff_0(X)$- quasi-invariant.

- Γ-gradient: for $v \in Vect_0(X)$, $F : \Gamma_X \to \mathbb{R}$ set

$$\begin{aligned} \nabla_v^\Gamma F(\gamma) \quad : \quad &= \frac{d}{dt} F(\varphi_t^v \gamma)_{t=0} \\ &= \sum_{x \in \gamma} (\nabla_x F(\gamma), v(x))_{T_x X} \end{aligned}$$

  and

$$\nabla^\Gamma F(\gamma) := (\nabla_x F(\gamma))_{x \in \gamma} \in \bigoplus_{x \in \gamma} T_x X$$

- Tangent space (Gelfand, Graev, Vershik 75):

$$T_\gamma \Gamma_X := \bigoplus_{x \in \gamma} T_x X$$

Integration by parts formula

Albeverio, Kondratiev, Röckner 96

$$\int_{\Gamma_X} \nabla_v^{\Gamma} F(\gamma)\pi(d\gamma) = -\int_{\Gamma_X} F(\gamma) B_\pi^v(\gamma)\pi(d\gamma),$$
$$B_\pi^v(\gamma) = \langle \beta_\sigma^v, \gamma \rangle$$

for local function F (that is $F(\gamma) = F(\gamma \cap \Lambda)$)

$B_\pi^v(\gamma)$ - logarithmic derivative of π along v,

β_σ^v - logarithmic derivative of σ along v ($= \mathrm{div} v$ if σ is Riemannian volume)

General notation: $\langle f, \gamma \rangle := \sum_{x\in\gamma} f(x), \quad f \in C_0(X)$

Laplace (Dirichlet) operator

Dirichlet form: for local functions F and G on Γ_X

$$\mathcal{E}_\pi(F,G) \quad := \int_{\Gamma_X} \left(\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\right)_{T_\gamma \Gamma_X} \pi(d\gamma)$$
$$\stackrel{IBP}{=} -\int_{\Gamma_X} \left(H_\pi^\Gamma F\right)(\gamma) G(\gamma) \pi(d\gamma)$$

Dirichlet operator $H_\pi^\Gamma F(\gamma) = \sum_{x\in\gamma} \left(\Delta_x + \beta_\sigma \nabla_x\right) F(\gamma)$- self-adjoint operator in $L^2(\Gamma_X, \pi)$

- H_π^Γ is essentially self-adjoint on smooth local functions
- $\mathrm{Ker} H_\pi^\Gamma = \{const\}$
- $L^2(\Gamma_X, \pi) \simeq \mathrm{Fock}(L^2(X,\sigma))$ and $H_\pi^\Gamma \simeq$ second quantization of $\Delta + \beta_\sigma \nabla$
- $\exp(-tH_\pi^\Gamma)$ - Markov semigroup, corresponding stochastic process is *independent Brownian motion of particles*

Where to go now?

$$\left(L^2(\Gamma_X,\pi),\ H^{\Gamma}_{\pi}\right)$$
Free Gas

$\swarrow$ $\searrow$

$$\left(L^2(\Gamma_X,\mu),\ H^{\Gamma}_{\mu}\right)$$
Interacting Gas
[AKR et al]

$$\left(L^2_{\pi}\Omega^p,\ H^{(p)}_{\pi}\right)$$
de Rham Complex

$\searrow$ $\swarrow$

$$\left(L^2_{\mu}\Omega^p,\ H^{(p)}_{\mu}\right)$$

μ is *Gibbs* or *cluster Gibbs* measure

Differential forms on configuration space

Reminder: $T_\gamma \Gamma_X = \bigoplus_{x\in\gamma} T_x X$

- 1-form (vector field) $\Gamma_X \ni \gamma \mapsto \omega(\gamma) \in T_\gamma \Gamma_X$
- p-form

$$\Gamma_X \ni \gamma \mapsto \omega(\gamma) \in \bigwedge^p T_\gamma \Gamma_X$$

- We work with square-integrable forms $\omega \in L^2_\pi \Omega^p$.
  Define Hodge differential

$$d_p : L^2_\pi \Omega^p \rightarrow L^2_\pi \Omega^{p+1}$$

by $d_p\omega(\gamma) = \text{anti-symmetrization } (\nabla^\Gamma \omega(\gamma))$.

$$d_p^* : L^2_\pi \Omega^{p+1} \rightarrow L^2_\pi \Omega^p$$

is densely defined (because of IBP)

- Hodge-de Rham Laplacian

$$\mathbb{H}^{(p)} := d_p d_p^* + d_{p-1}^* d_{p-1},$$

self-adjoint in $L^2_\pi \Omega^p$

Decomposition of Hodge-de Rham Laplacian

Albeverio, AD, Lytvynov:

1. $\exists$ isometry $\mathcal{I}^p : L^2_\pi\Omega^p \to L^2(\Gamma_X, \pi) \otimes \left[\bigoplus_{m=1}^p L^2_\sigma \Omega^p(X^m)\right]$
2. $$\mathbf{H}_\pi^{(p)} \overset{\mathcal{I}^p}{\simeq} H_\pi^\Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \left[\sum_{m=1}^p H_{X^m}^{(p)}\right],$$
   where $H_{X^m}^{(p)}$ is Hodge-de Rham on X^m
3. Künneth formula
   $$\mathrm{Ker}\ \mathbf{H}_\pi^{(p)} \overset{\mathcal{I}^p}{\simeq} \bigoplus_{s_1,\dots,s_d} \left(\mathcal{H}_X^{(1)}\right)^{\overset{1}{\Diamond} s_1} \otimes \dots \otimes \left(\mathcal{H}_X^{(d)}\right)^{\overset{d}{\Diamond} s_d},$$
   $\mathcal{H}_X^{(m)} := \mathrm{Ker}\ H_X^{(m)}$, $d := \dim X$,
   $$\overset{m}{\Diamond} s = \begin{cases} \widehat{\bigotimes}, & m \text{ even} \\ \bigwedge, & m \text{ odd} \end{cases}$$

Corollary: let $\beta_m := \dim \mathcal{H}_X^{(m)} < \infty$. Then

$$\dim \operatorname{Ker} \mathbf{H}_\pi^{(p)} = \sum_{s_1,\dots,s_d} \beta_1^{(s_1)} \dots \beta_d^{(s_d)},$$

$$\beta_m^{(s)} = \begin{pmatrix} \beta_m \\ s \end{pmatrix}, \ m \text{ odd, and } \beta_m^{(s)} = \begin{pmatrix} \beta_m + s - 1 \\ s \end{pmatrix}, \ m \text{ even.}$$

Example: manifold with cylinder end $X = M \cup \left(N \times \mathbb{R}^1\right)$, M compact with boundary N.

If $d = 2$ then $\beta_0 = \beta_2 = 0$, $\beta_1 < \infty$. Then

$$\dim \operatorname{Ker} \mathbf{H}_\pi^{(p)} = \begin{cases} \begin{pmatrix} \beta_1 \\ p \end{pmatrix} & \text{if } p \le \beta_1 \\ 0 & \text{if } p > \beta_1 \end{cases}$$

Remark: In general, $\dim \mathcal{H}_X^{(m)} = \infty$ because X is not compact

L2-Betti numbers (M. Atiyah 76)

Assumption: $\exists\ G \subset \mathrm{Iso}\ X$, infinite discrete; $M = X/G$- compact manifold. **Example**: $X = \mathbb{H}^d$-hyperbolic space

$$\text{group action} \qquad G \in g \mapsto T_g \in \mathcal{B}(L^2\Omega^p(X)).$$

Commutant of G-action

$$\mathcal{A} := \{T_g,\ g \in G\}' \subset \mathcal{B}(L^2\Omega^p(X)) -$$

von Neumann algebra (II_∞ factor if G is "strongly" non-commutative).
Orthogonal projection

$$P : L^2\Omega^p(X) \to \mathcal{H}_X^{(p)}.$$

- $P \in \mathcal{A}$, L^2-Betti number $b_n = \dim_G \mathcal{H}_X^{(p)} := \mathrm{Tr}_{\mathcal{A}} P < \infty$
- $b_n = 0$ iff $\dim\ \mathcal{H}_X^{(m)} < \infty$
- $\mathrm{Index}_{\mathcal{A}}(d + d^*) := \sum(-1)^p b_p = \chi(M)$- Euler characteristic of M

Traces of projections in tensor products of factors

$\mathcal{H} \subset \mathfrak{X}$ - Hilbert spaces; $P : \mathfrak{X} \to \mathcal{H}$ - orthogonal projection; assume $P \in \mathcal{A}$ - some von Neumann algebra $\neq \mathcal{B}(\mathcal{H})$, $\mathrm{Tr}_{\mathcal{A}} P < \infty$.
Define

$$P_s^{(n)} : \mathfrak{X}^{\otimes n} \to \mathcal{H}^{\widehat{\otimes} n}, \quad P_a^{(n)} : \mathfrak{X}^{\otimes n} \to \mathcal{H}^{\wedge n}.$$

$P_s^{(n)}, P_a^{(n)} \in \{\mathcal{A}^{\otimes n}, \{U_\alpha, \alpha \in S_n\}\}'' := \mathcal{A}^{(n)}$ ($\neq \mathcal{A}^{\otimes n}$ in general).

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Theorem

1) $\mathcal{A}$ - II_1*-factor. Then*

$$\begin{aligned} \mathcal{A}^{(n)} &= W^*(\mathcal{A}^{\otimes n}, S_n) \quad \textit{(cross-product);} \\ \mathrm{Tr}_{\mathcal{A}} P_s^{(n)} &= \mathrm{Tr}_{\mathcal{A}} P_a^{(n)} = \frac{(\mathrm{Tr}_{\mathcal{A}} P)^n}{n!}. \end{aligned}$$

2) $\mathcal{A}$ - II_∞*-factor,* $\mathcal{A} = \mathcal{B}(\mathbb{C}^m) \otimes \mathcal{M}$, $\mathcal{M}$ - II_1*-factor. Then*

$$\mathcal{A}^{(n)} = \mathcal{B}(\mathbb{C}^m)^{\otimes n} \otimes W^*(\mathcal{M}^{\otimes n}, S_n).$$

$$b_p(\Gamma_X) := \dim_G \ \mathrm{Ker} \ \mathbf{H}_\pi^{(p)} = \sum_{s_1,\dots,s_d} \frac{b_1^{s_1}}{s_1!} \dots \frac{b_d^{s_d}}{s_d!}, \qquad d = \dim X - 1$$

Example. $X = \mathbb{H}^{d+1}$ - hyperbolic space,

$$b_p(X) = \begin{cases} 0, & p \neq \frac{1}{2}(d+1) \\ c, & p = \frac{1}{2}(d+1) \end{cases}, \quad p = 1, 2, \dots, d.$$

Then

$$b_p(\Gamma_X) = \begin{cases} 0, & p \neq \frac{1}{2}(d+1)s \\ \frac{c^s}{s!}, & p = \frac{1}{2}(d+1)s \end{cases}, \quad s = 1, 2, \dots$$

Index of the Dirac operator

$$D := d + d^{*:} : L^2_\pi \Omega^{\mathrm{even}} \to L^2_\pi \Omega^{\mathrm{odd}}$$

$$\begin{aligned} \mathrm{ind}_G &= \dim_G \ \mathrm{Ker} \ D - \dim_G \ \mathrm{Ker} \ D^* \\ &= \sum_p (-1)^p b_p(\Gamma_X) = \exp \chi(X/G). \end{aligned}$$

Gibbs measures on configuration spaces

$v : \mathbb{R} \to \mathbb{R}$; ρ - distance in X; pair potential $V(x,y) = v(\rho(x,y))$;

$$\text{Energy} \quad E(\gamma) = \sum_{x,y\in\gamma} V(x,y), \ \gamma \in \Gamma_X$$

$$\text{Gibbs measure} \quad \mu(d\gamma) = \text{"}\frac{1}{Z}\exp\left(-E(\gamma)\right)\pi(d\gamma)\text{"}.$$

- μ - probability measure on Γ_X - well-defined (*conditions on potential* V);
- in general, μ is not unique (*phase transitions*);
- μ is $Diff_0(X)$-quasi-invariant;
- μ-symmetric Hodge-de Rham Laplacian in $L^2(\Gamma_X, \mu)$ has *complicated structure.*

Random Witten Laplacian on X

Define measure σ_γ $(\gamma \in \Gamma_X)$ on X :

$$\sigma_\gamma(dx) = \exp\left(-E_\gamma(x)\right) dx, \text{ where } E_\gamma(x) = \sum_{y\in\gamma} V(x,y).$$

Witten Laplacian $H^{(p)}_{\sigma_\gamma}$ associated with σ_γ:
symmetrization of the classical Hodge-de Rham $H^{(p)}$ in $L^2_{\sigma_\gamma}\Omega^p(X)$.

$$\begin{aligned} L^2_{\sigma_\gamma}\Omega^p(X) &\sim L^2\Omega^p(X) \\ H^{(p)}_{\sigma_\gamma} &\sim H^{(p)}_\gamma = H^{(p)} + W^{(p)}_\gamma, \end{aligned}$$

where

$$\begin{aligned} W^{(p)}_\gamma(x) &: (T_xX)^{\wedge p} \to (T_xX)^{\wedge p}, \\ W^{(p)}_\gamma(x) &= \left\|\nabla E_\gamma(x)\right\|^2 \mathrm{id} + \Delta E_\gamma(x)\mathrm{id} + \left(\nabla^2 E_\gamma(x)\right)^{\wedge p}. \end{aligned}$$

$H^{(p)}_\gamma$ is self-adjoint positive operator in $L^2\Omega^p(X)$

Trace of the semigroup

Goal: to define (von Neumann) trace of the semigroup $e^{-tH_\gamma^{(p)}}$
Framework: $G \subset \operatorname{Iso} X$, infinite discrete; $M = X/G$ - compact manifold; σ_γ is G-invariant
Action of G on Γ_X:

$$\gamma = \{..., x, y, z, ...\} \mapsto g\gamma = \{..., gx, gy, gz, ...\}, \; g \in G;$$

T_g - associated (unitary) representation of G in $L^2\Omega^p(X)$;

$$T_g H_\gamma^{(p)} T_g^{-1} = H_{g\gamma}^{(p)}.$$

Thus $e^{-tH_\gamma^{(p)}} \notin \mathcal{A} := \{T_g, \; g \in G\}'$ for μ-a.a. $\gamma \in \Gamma_X$
(since $\mu(\{\gamma : \; g\gamma = \gamma \; \forall g \in G\}) = 0$).

Define

$$\begin{aligned}
\mathcal{C} &= \{A : \Gamma_X \to \mathcal{B}(L^2\Omega^*(X)), \text{ bounded}, \; A(g\gamma) = T_g A(\gamma) T_g^{-1}\}, \\
\omega(A) &= \int_{\Gamma_X} \int_{X/G} a_\gamma(x,x)dx \; \mu(d\gamma), \; a_\gamma \text{ - integral kernel of } A.
\end{aligned}$$

Theorem

$\mathcal{C}$ is a von Neumann algebra, ω is a faithful normal semifinite trace on $\mathcal{C}$.

$\mathbf{P}_\gamma^{(p)}$ - ortho projection onto $\mathrm{Ker}\ H_\gamma^{(p)}$. Consider maps $\mathbf{P}^{(p)} : \gamma \mapsto \mathbf{P}_\gamma^{(p)}$, $e^{-tH^{(p)}} : \gamma \mapsto e^{-tH_\gamma^{(p)}}$. Then $\mathbf{P}^{(p)}, e^{-tH^{(p)}} \in \mathcal{C}$.

Theorem

1)

$$\omega\left(\mathbf{P}^{(p)}\right) < \infty,\ \omega\left(e^{-tH^{(p)}}\right) < \infty$$

2) McKean-Singer formula

$$\mathrm{STR}\ e^{-tH} := \sum_p (-1)^p \omega\left(e^{-tH_\gamma^{(p)}}\right) = \mathrm{STR}\ \mathbf{P}$$

3)

$$\mathrm{STR}\ \mathbf{P} = \chi(X/G) \text{ for any Gibbs measure } \mu$$

Proof is based on the probabilistic representation of $e^{-tH^{(p)}}$, McKean-Singer formula in general von Neumann algebras (Connes-Moscovichi) and study of short-time asymptotics of STR e^{-tH}.

Corollary

Let $\chi(X/G) \neq 0$ *and* μ *be* G*-ergodic.*
Then

$$\dim \text{Ker } H_\gamma = \infty \text{ for } \mu\text{-a.a. } \gamma \in \Gamma_X.$$

Open questions

- to compute individual Betti numbers
- are there measures for which the index is different?
- to consider the full Hodge-de Rham operator associated with μ on Γ_X

Recent developments- analysis of cluster measures

> *"Between isolated atoms or molecules and bulk materials there lies a class of unique structures, known as clusters, that consist of a few to hundreds of atoms or molecules. Within this range of "nanophase" ... materials may exhibit novel properties due to quantum confinement effects."*

Dynamics of clusters: From elementary to biological structures. Po-Y. Cheng, J.S. Baskin, A.H. Zewail (Arthur Amos Noyes Laboratory of Chemical Physics, California Institute of Technology) PNAS, 2006

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What is a Cluster Point Process (CPP)? (e.g. Daley, Vere-Jones)

- Take a random configuration $\gamma_c \subset X$ of *invisible* centres.
- To each $x \in \gamma_c$, attach a set $Z_x = \left\{ z_j^{(x)} \right\}_{j=1}^{N}$ of *observable* points (*cluster* at x).
- Resulting configuration $Z = \cup_{x \in \gamma_c} Z_x$ is a *Cluster Point Process.*

Cluster measure - distribution of CPP

Construction of cluster measures on Γ_X (L. Bogachev, AD)

1. "Unpacking" mapping $\mathfrak{p} : \Gamma_{X^n} \to \Gamma_X$

$$X^n \ni \bar{x} = (x_1, ...x_n) \overset{p}{\mapsto} \{x_1, ...x_n\} \subset X,$$
$$\Gamma_{X^n} \ni \{..., \bar{x}, \bar{y}, \bar{z}, ...\} \overset{\mathfrak{p}}{\mapsto} \{....x_1, ...x_n, y_1, ...y_n, z_1, ...z_n, ...\} \in \Gamma_X.$$

2. Poisson or Gibbs measure μ on Γ_{X^n}.
3. Cluster measure $\mu_{cl} := p^*\mu$:

$$\mu_{cl}(A) := \mu(\mathfrak{p}(A)).$$