Representation categories of quantum groups

Leonid Vainerman

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Abstract

We start with the basic notions related to tensor categories and functors. The most important example is the representation category of a quantum group. We discuss braided tensor categories and such important constructions as the center of a tensor category and the Drinfeld's double of a finite group. Finally, we consider ribbon categories and ribbon Hopf algebras.

Prerequisits: algebras and modules, tensor product of vector spaces, representation of groups and Hopf algebras.

1 Lecture 1. Tensor categories. Braiding.

1) Categories and functors.

In what follows k denotes an algebraically closed field with $char(k) = 0$. The most important concrete example is $k = \mathbb{C}$.

Definition 1.1 *A category $\mathcal{C}$ consists*

(1) of a class $Ob(\mathcal{C})$ whose elements are called **objects** *of $\mathcal{C}$,*

(2) of a class $Hom(\mathcal{C})$ whose elements are called **morphisms** *of $\mathcal{C}$,*

(3) of maps: identity $id : Ob(\mathcal{C}) \to Hom(\mathcal{C})$, source $s : Hom(\mathcal{C}) \to Ob(\mathcal{C})$, target $b : Hom(\mathcal{C}) \to Ob(\mathcal{C})$, composition $\circ : Hom(\mathcal{C}) \times_{Ob(\mathcal{C})} Hom(\mathcal{C}) \to Hom(\mathcal{C})$ such that

$$s(id_V) = b(id_V) = V, \quad id_{b(f)} \circ f = f \circ id_{s(f)} = f \text{ for all } V \in Ob(\mathcal{C}),\ f \in Hom(\mathcal{C})$$

and $(h \circ g) \circ f = h \circ (g \circ f)$ for all $f, g, h \in Hom(\mathcal{C})$ satisfying $b(f) = s(g)$ and $b(g) = s(h)$. Here $Hom(\mathcal{C}) \times_{Ob(\mathcal{C})} Hom(\mathcal{C})$ denotes the class of couples (f, g) of **composable** *morphisms of $\mathcal{C}$, i.e., such that $b(f) = s(g)$. We denote by $g \circ f$ the composition of f and g, and by $Hom_{\mathcal{C}}(V, W)$ the class of morphisms of $\mathcal{C}$ whose source is V and target is W $(V, W \in Ob(\mathcal{C}))$. For $f \in Hom_{\mathcal{C}}(V, W)$ we write $f : V \to W$. A morphism $f : V \to V$ is called an* **endomorphism** *of V, the class of such morphisms is denoted by $End(V)$. A morphism $f : V \to W$ is called an* **isomorphism** *if there is a morphism $g : W \to V$ such that $g \circ f = id_V$, $f \circ g = id_W$.*

Example 1.2 *Categories:* **Set** *of sets,* **Gr** *of groups,* $Vec(k)$ *of vector spaces,* $Vec_f(k)$ *of finite-dimensional vector spaces,* **Alg** *of associative algebras over* k*. Given an algebra* A*, we denote by* $Mod(A)$ *the category whose objects are left* A*-modules and morphisms are* A*-linear maps. More examples: the category of* W^**-algebras whose morphisms are normal homomorphisms, the category of Hopf-*W^**-algebras whose morphisms are normal homomorphisms of* W^**-algebras such that* $\Delta \circ f = (f \otimes f) \circ \Delta$.

The product $\mathcal{C}\times\mathcal{D}$ of two categories is the category whose objects are pairs of objects $(V, W) \in \mathcal{C} \times \mathcal{D}$ and morphisms are given by $Hom_{\mathcal{C}\times\mathcal{D}}((V, W), (V', W')) = Hom_{\mathcal{C}}(V, V') \times Hom_{\mathcal{D}}(W, W')$. A **subcategory** $\mathcal{C}$ of a category $\mathcal{D}$ consists of a subclass $Ob(\mathcal{C})$ of $Ob(\mathcal{D})$ and of a subclass $Hom(\mathcal{C})$ of $Hom(\mathcal{D})$ that are stable under the identity, source, target and the composition maps in $\mathcal{D}$.

Definition 1.3 *A* **functor** $F : \mathcal{C} \to \mathcal{D}$ *between two categories consists of a map* $F : Ob(\mathcal{C}) \to Ob(\mathcal{C}')$ *and of a map* $F : Hom(\mathcal{C}) \to Hom(\mathcal{D})$ *such that*

(a) $F(id_V) = id_{F(V)}$ *for any* $V \in Ob(\mathcal{C})$*,*

(b) $s(F(f)) = F(s(f))$ *and* $b(F(f)) = F(b(f))$ *for any* $f \in Hom(\mathcal{C})$*,*

(c) $F(g \circ f) = F(g) \circ F(f)$ *for any composable morphisms in* $\mathcal{C}$*.*

A functor $F : \mathcal{C} \to \mathcal{D}$ *is called* **essentially surjective** *if, for any* $W \in Ob(\mathcal{D})$*, there is* $V \in Ob(\mathcal{C})$ *such that* $F(V)$ *is isomorphic to* W *in* $\mathcal{D}$*.* F *is called* **faithful** *(resp.,* **fully faithful***) if, for any* $V, V' \in Ob(\mathcal{C})$*, the map* $F : Hom_{\mathcal{C}}(V, V') \to Hom_{\mathcal{D}}(F(V), F(V'))$ *on morphisms is injective (resp., bijective).*

The composition of two functors is a functor, for any $\mathcal{C}$ there is a functor $id_{\mathcal{C}}$, the inclusion of a subcategory in a category is a functor.

Definition 1.4 *A natural transformation* η *from* $F : \mathcal{C} \to \mathcal{C}'$ *to* $G : \mathcal{C} \to \mathcal{C}'$ *(we write* $\eta : F \to G$*) is a family of morphisms* $\eta(V) : F(V) \to G(V)$ *in* $\mathcal{C}'$ *(*$V \in Ob(\mathcal{C})$*) such that, for any morphism* $f : V \to W$ *in* $\mathcal{C}$*, we have* $G(f) \circ \eta(V) = \eta(W) \circ F(f)$*. If, in particular, all of* $\eta(V)$ *are isomorphisms, we say that* $\eta : F \to G$ *is a natural isomorphism (in this case* $\eta(V)^{-1}$ *defines a natural isomorphism* $\eta^{-1} : G \to F$*).*

Definition 1.5 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called an equivalence of categories if it is essentially surjective and fully faithful.*

2) Tensor (or monoidal) categories and functors.

A **tensor product** on a category $\mathcal{C}$ is functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. This means that, for any pairs $V, W \in Ob(\mathcal{C})$, $f, g \in Hom(\mathcal{C})$, there are an object $V \otimes W \in Ob(\mathcal{C})$ and a morphism $f \otimes g \in Hom(\mathcal{C})$ such that $s(f \otimes g) = s(f) \otimes s(g)$, $b(f\otimes g) = b(f)\otimes s(g)$, $id_{V\otimes W} = id_V \otimes id_W$ and $(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g)$ for any pairs of composable morphisms (f, f') and (g, g').

An **associativity constraint** for $\otimes$ is a natural isomorphism $a : \otimes(\otimes\times id) \to \otimes(id \times \otimes)$. This means that, for any $U, V, W \in Ob(\mathcal{C})$, there is an isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that $[(f \otimes (g \otimes h]] \circ a_{U,V,W} =$

$a_{U',V',W'} \circ [(f \otimes g) \otimes h]$ for any morphisms $f : U \to U', g : V \to V'$ and $h : W \to W'$ in $\mathcal{C}$. This isomorphism should also verify the **Pentagon axiom**:

$$\begin{array}{ccc}
(U \otimes (V \otimes W)) \otimes X & \xleftarrow{a_{U,V,W} \otimes id_X} & ((U \otimes V) \otimes W) \otimes X \\
 & & \downarrow a_{U \otimes V,W,X} \\
\downarrow a_{U,V \otimes W,X} & & (U \otimes V) \otimes (W \otimes X) \\
 & & \downarrow a_{U,V,W \otimes X} \\
U \otimes ((V \otimes W) \otimes X & \xrightarrow{id_U \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X))
\end{array}$$

- this diagram commutes for all objects U, V, W, X of $\mathcal{C}$.

A **left (resp., right) unit constraint** with respect to a fixed $I \in Ob(\mathcal{C})$ is a natural isomorphism $l : \otimes(I \otimes id) \to id$ (resp., $r : \otimes(id \otimes I) \to id$. This means that, for any $V \in Ob(\mathcal{C})$, there is an isomorphism $l_V : I \otimes V \to V$ (resp., $r_V : V \otimes I \to V$) such that $f \circ l_V = l_{V'}(id_I \otimes f)$ (resp., $f \circ r_V = r_{V'}(f \otimes id_I)$) for any morphism $f : V \to V'$. The associativity, left and right unit constraints should also verify the **Triangle axiom**:

$$r_V \otimes id_W = (id_V \otimes l_W) \circ a_{V,I,W} \quad \text{for all objects} \quad V, W.$$

Definition 1.6 *A tensor category $(\mathcal{C}, \otimes, a, l, r)$ is a category $\mathcal{C}$ equipped with a tensor product $\otimes$, with an associativity constraint a, with a fixed object I (called the unit of a tensor category), with left and right unit constraints l and r with respect to I satisfying the Pentagon and the Triangle axioms. It is said to be* **strict** *if α, l, r are all identities.*

Example 1.7 *1. $\mathcal{C} = Vec(k)$ with usual tensor product of vector spaces, $I = k$, $a((u \otimes v) \otimes w) = u \otimes (v \otimes w)$, $l(1 \otimes v) = v = r(1 \otimes v)$ for all $v \in V, w \in W$, V, W - arbitrary vector spaces. The category $Vec_f(k)$ of finite-dimensional vector spaces is a subcategory of $Vec(k)$ with the same $\otimes, a, l, r$ (a tensor subcategory).*

2. $\mathcal{C} = Rep(G)$ - a tensor subcategory of $Vec(k)$ whose objects are G-modules (equivalently - kG-modules), where G-action $g \cdot (u \otimes v) = (g \cdot u) \otimes (g \cdot v)$, $g \cdot \lambda = \lambda$ for all $g \in G, u \in U, v \in V, \lambda \in k$, U, V - G-modules. Morphisms - G-linear maps of G-modules.

3. More generally, let A be an associative unital k-algebra equipped with morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ of unital algebras. Let $Mod(A)$ be a category of left A-modules (i.e., representations of A). If U, V are two left A-modules, then $U \otimes V$ becomes a left A-module by $a \cdot (u \otimes v) = \Delta(a) \cdot (u \otimes v)$ for all $a \in A, u \in U, v \in V$. k is a left A-module by $a \cdot \lambda = \varepsilon(a)\lambda$. Morphisms - A-linear maps of A-modules.

It is clear that $\otimes$ in $Vec(k)$ restricts to a functor $\otimes : Mod(A) \times Mod(A) \to Mod(A)$ for which $I = k$ is a unit. Then we have

Proposition 1.8 *Let (A, Δ, ε) be a triple as above. It is a* **bialgebra** *(i.e., $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, $(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$) iff $Mod(A)$ is a tensor subcategory of $Vec(k)$ (i.e., with the same $\otimes, a, l, r$).*

Proof. *(i)* **Exercise.** *Let $(A, \varphi, \eta, \Delta, \varepsilon)$ be a bialgebra and U, V, W be left A-modules. Check that the canonical isomorphisms of vector spaces $a_{U,V,W}$:*

$(U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l_V : k \otimes V \to V$ and $r_V : V \otimes k \to V$ are left A-module morphisms.

(ii) Conversely, let $Mod(A)$ be a tensor subcategory of $Vec(k)$. The A-linearity of $a_{A,A,A}$ means that, for all $b, u, v, w \in A$:

$$a_{A,A,A}(b \cdot [(u \otimes v) \otimes w]) = b \cdot a_{A,A,A}[(u \otimes v) \otimes w].$$

By definition of $a_{A,A,A}$, this can be rewritten as

$$((\Delta \otimes id)\Delta(b)) \cdot [u \otimes (v \otimes w)] = (id \otimes \Delta)\Delta(b) \cdot [u \otimes (v \otimes w)].$$

For $u, v, w = 1_A$, we get the coassociativity of Δ. Similarly, l_A and r_A are A-linear iff $(\varepsilon \otimes id)\Delta(b) = b$ (resp., $(id \otimes \varepsilon)\Delta(b) = b$) for all $b \in A$. □

In what follows we will denote $Rep(A) = (Mod(A), \otimes)$.

Remark 1.9 *$Mod(A)$ is a tensor category (not necessarily strict) iff (A, Δ, ε) is a* **quasi-bialgebra** *- see [1].*

Example 1.10 *of a non strict tensor category.*

Consider the strict tensor category $\mathcal{C} = Rep(A)$, where $(A = Fun(G), \Delta, \varepsilon)$ is the bialgebra associated with a finite group G and change the associativity constraint. Since A is semisimple, any left A-module is completely reducible, so in order to define a morphism $f : V \to W$, it suffices to define it only for irreducible components of V and W (such categories are called semisimple). But all irreducible A-modules are 1-dimensional and are parameterized by the elements of G : $f \cdot V_g = f(g)V_g$, and the only nontrivial morphisms between them are of the form $\lambda\ id_{V_g}$, where $\lambda \in k$, $g \in G$. Since $\Delta(f)(g,h) := f(gh)$, $\varepsilon(f) = f(e)$ for all $f \in Fun(G)$, $g, h \in G$, then $V_g \otimes V_h = V_{gh}$, $I = V_e$, where e is the unit of G. Thus, in order to study possible associativity constraints in $\mathcal{C}$, it suffices to study the Pentagon axiom for irreducibles parameterized by $g, h, k, l \in G$.

First, we see that $a_{V_g,V_h,V_k} : (V_g \otimes V_h) \otimes V_k \to V_g \otimes (V_h \otimes V_k)$ must be of the form $a_{V_g,V_h,V_k} = \omega(g,h,k) id_{V_{ghk}}$, where $\omega : G \times G \times G \to k^\times$ is a scalar function. Second, the Pentagon axiom is equivalent to

$$\omega(g,h,kl)\omega(gh,k,l)\omega(g,h,k) = \omega(h,k,l)\omega(g,hk,l) \quad \text{for all} \quad g,h,k,l \in G,$$

-the 3-cocycle equation. Thus, taking nontrivial 3-cocycles on G, we get various structures of non strict tensor category on $Mod(Fun(G))$.

Definition 1.11 *(a) Let $(\mathcal{C}, \otimes, I_\mathcal{C}, a, l, r)$ and $(\mathcal{D}, \otimes, I_\mathcal{D}, a, l, r)$ be tensor categories. A* **tensor functor** *from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$, where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\varphi_0 : I_\mathcal{D} \to F(I_\mathcal{C},)$ is an isomorphism, and $\varphi_2(U,V) : F(U) \otimes F(V) \to F(U \otimes V)$ is a family of natural isomorphisms indexed by all couples of objects of $\mathcal{C}$ such that the diagrams*

$$\begin{array}{ccc}
(F(U) \otimes F(V)) \otimes F(V) & \xrightarrow{a_{F(U),F(V),F(W)}} & F(U) \otimes (F(V) \otimes F(W)) \\
\downarrow {\scriptstyle \varphi_2(U,V) \otimes id_{F(W)}} & & \downarrow {\scriptstyle id_{F(U)} \otimes \varphi_2(V,W)} \\
F(U \otimes V) \otimes F(W) & & F(U) \otimes F(V \otimes W) \\
\downarrow {\scriptstyle \varphi_2(U \otimes V, W)} & & \downarrow {\scriptstyle \varphi_2(U, V \otimes W)} \\
F((U \otimes V) \otimes W) & \xrightarrow{F(a_{U,V,W})} & F(U \otimes (V \otimes W))
\end{array}$$

$$\begin{CD}
I_{\mathcal{D}} \otimes F(U) @>{l_{F(U)}}>> F(U) \\
@V{\varphi_0 \otimes id_{F(U)}}VV @AA{F(l_U)}A \\
F(I_{\mathcal{C}}) \otimes F(U) @>{\varphi_2(I_{\mathcal{C}},U)}>> F(I_{\mathcal{C}} \otimes U)
\end{CD}$$

$$\begin{CD}
F(U) \otimes I_{\mathcal{D}} @>{r_{F(U)}}>> F(U) \\
@V{id_{F(U)\otimes\varphi_0}}VV @AA{F(r_U)}A \\
F(U) \otimes F(I_{\mathcal{C}}) @>{\varphi_2(U,I_{\mathcal{C}})}>> F(U \otimes I_{\mathcal{C}})
\end{CD}$$

commute for all objects U, V, W of $\mathcal{C}$. It is said to be **strict** *if φ_0 and φ_2 are identities of $\mathcal{D}$.*

(b) A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ of tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$ such that the following diagrams commute for all couples (U, V) of objects of $\mathcal{C}$:

$$\begin{CD}
F(U) \otimes F(V) @>{\varphi_2(U,V)}>> F(U \otimes V) \\
@V{\eta(U)\otimes\eta(V)}VV @VV{\eta(U\otimes V)}V \\
F'(U) \otimes F'(V) @>{\varphi_2'(U,V)}>> F'(U \otimes V)
\end{CD}$$

and $\varphi_0' = \eta(I_{\mathcal{C}}) \circ \varphi_0$. A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.

c) A tensor equivalence of tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a tensor functor $F' : \mathcal{D} \to \mathcal{C}$ and a natural tensor isomorphisms $\eta : id_{\mathcal{D}} \to F \circ F'$ and $\theta : F' \circ F \to id_{\mathcal{C}}$.

A composition of tensor functors is again a tensor functor, and the identity functor is a strict tensor functor.

Example 1.12 *1. Let A be a bialgebra. The forgetful functor associating to an A-module its underlying vector space is a strict tensor functor from $Rep(A)$ to $Vec(k)$.*

2. Let $f : A_1 \to A_2$ be a morphism of bialgebras. We can equip any A_2-module V with an A_1-module structure by $a \cdot v := f(a) \cdot v$ for all $a \in A_1, v \in V$. This gives a strict tensor functor $f^* : Rep(A_2) \to Rep(A_1)$.

Remark 1.13 *One can show (see [1]) that any tensor category is tensor equivalent to a strict tensor category.*

3) Braided tensor categories and functors.

Definition 1.14 *a) A braiding in a tensor category $(\mathcal{C}, \otimes, a, l, r)$ is a natural isomorphism $c : \otimes \to \otimes \circ \tau$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the flip functor defined by $\tau(V, W) = (W, V)$ on any pair of objects of $\mathcal{C}$, i.e., a family of isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ defined for any couple (V, W) of objects of $\mathcal{C}$ such that,* *for any morphisms $f : V \to V'$ and $g : W \to W'$, the square*

$$\begin{array}{ccc} V\otimes W & \xrightarrow{\quad c_{V,W}\quad} & W\otimes V \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ V'\otimes W' & \xrightarrow{\quad c_{V',W'}\quad} & W'\otimes V' \end{array}$$

commutes and satisfies the **Hexagon axioms**, *i.e., the diagrams*

$$\begin{array}{ccc} U\otimes(V\otimes W) & \xrightarrow{\quad c_{U,V\otimes W}\quad} & (V\otimes W)\otimes U \\ \uparrow{\scriptstyle a_{U,V,W}} & & \downarrow{\scriptstyle a_{V,W,U}} \\ (U\otimes V)\otimes W & & V\otimes(W\otimes U) \\ {\scriptstyle c_{U,V}\otimes id_W}\downarrow & & {\scriptstyle id_V\otimes c_{U,W}}\uparrow \\ (V\otimes U)\otimes W & \xrightarrow{\quad a_{V,U,W}\quad} & V\otimes(U\otimes W) \end{array}$$

$$\begin{array}{ccc} (U\otimes V)\otimes W & \xrightarrow{\quad c_{U\otimes V,W}\quad} & W\otimes(U\otimes V) \\ \uparrow{\scriptstyle a^{-1}_{U,V,W}} & & \downarrow{\scriptstyle a^{-1}_{W,U,V}} \\ U\otimes(V\otimes W) & & (W\otimes U)\otimes V \\ {\scriptstyle id_U\otimes c_{V,W}}\downarrow & & {\scriptstyle c_{U,W}\otimes id_V}\uparrow \\ U\otimes(W\otimes V) & \xrightarrow{\quad a^{-1}_{U,W,V}\quad} & (U\otimes W)\otimes V \end{array}$$

commute for all objects U, V, W of $\mathcal{C}$.

b) A braided tensor category $(\mathcal{C}, \otimes, a, l, r, c)$ is a tensor category with braiding.

Remark that if c is a braiding, then so is c^{-1}. In a strict tensor category the above diagrams are equivalent, respectively, to

$c_{U,V\otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)$ and $c_{U\otimes V,W} = (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W})$,

from where, in particular, $c_{I,I} = id_I$.

Example 1.15 *1. The usual tensor flip τ of vector spaces is a braiding in $Vec(k)$ and in $Rep(G)$.*

2. Braiding in the category of representations of a bialgebra.

Definition 1.16 *Let (A, Δ, ε) be a bialgebra. An invertible element $R = \sum a_i \otimes b_i = R_{(1)} \otimes R_{(2)} \in A \otimes A$ is called a* **universal R-matrix** *if it satisfies*

$$\Delta^{op}(a) = R\Delta(a)R^{-1}, \quad (id \otimes \Delta)R = R_{13}R_{12}, \quad (\Delta \otimes id)R = R_{13}R_{23},$$

where $a \in A$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_{13} = \sum a_i \otimes 1 \otimes b_i$. A bialgebra (resp., Hopf algebra) possessing a universal R-matrix is called braided or quasi-triangular.

Exercises. 1. Show that a universal R-matrix verifies $(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1_A$.

Hint: Apply $id \otimes \varepsilon \otimes id$ to the two last equalities of the definition of a universal R-matrix.

2. Show that a universal R-matrix verifies $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ - the **quantum Yang-Baxter equation**.

3. Let $(A, \Delta, \varepsilon, S)$ be a braided Hopf algebra with invertible antipode S and with a universal R-matrix R. Using Exercises 1 and 2, and relations $m(S \otimes id_A)\Delta(a) = m(id_A \otimes S)\Delta(a) = m(S^{-1} \otimes id_A)\Delta^{op}(a) = m(id_A \otimes S^{-1})\Delta^{op}(a) = \varepsilon(a)1$ (for all $a \in A$), show that $R^{-1} = (S \otimes id_A)(R) = (id_A \otimes S^{-1})(R)$.

Proposition 1.17 *A bialgebra (A, Δ, ε) is braided iff the strict tensor category $Rep(A)$ is braided.*

Proof. a) Let R be a universal R-matrix for A. Let us define isomorphisms $c^R_{V,W} : V \otimes W \to W \otimes V$ by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)) \quad \text{for all} \quad v \in V, w \in W.$$

Its inverse is given by $(c^R_{V,W})^{-1}(w \otimes v) = R^{-1}(v \otimes w)$ from where $(c^R_{V,W})^{-1} \circ \tau_{V,W}(v \otimes w) = R^{-1}(v \otimes w)$.

Now let us check that the axioms for R are equivalent to the requirement that $c_{V,W}$ is a braiding. First, $c_{V,W}$ is A-linear:

$$a \cdot c^R_{V,W}(v \otimes w) = \Delta(a) \cdot \tau_{V,W}(R(v \otimes w)) = \tau_{V,W}(\Delta^{op}(a)R(v \otimes w)) =$$

$$= \tau_{V,W}(R\Delta(a)(v \otimes w)) = c_{V,W}(a \cdot (v \otimes w)).$$

Then

$$(id_V \otimes c^R_{U,W})(c^R_{U,V} \otimes id_W)(u \otimes v \otimes w) = R_{(2)}v \otimes R'_{(2)}w \otimes R'_{(1)}R_{(1)}u =$$

$$= \Delta(R_{(2)}) \cdot (v \otimes w) \otimes R_{(1)}u = c_{U,V\otimes W}(u \otimes v \otimes w)$$

because $(id \otimes \Delta)(R) = R_{13}R_{12} = R'_{(1)}R_{(1)} \otimes R_{(2)} \otimes R'_{(2)}$. Similarly one can check the remaining relation for $c_{V,W}$.

b) Let c be a braiding in $Rep(A)$, where (A, Δ, ε) is a bialgebra. Let us show that an invertible element $R := \tau_{A,A}(c_{A,A}(1 \otimes 1)$ is a universal R-matrix. For any $v \in V, w \in W$, where V and W are A-modules, define A-linear maps $\alpha_v : A \to V$ and $\alpha_w : A \to W$ by $\alpha_v(1) = v, \alpha_w(1) = w$, then the naturality of c implies that $(\alpha_w \otimes \alpha_v) \circ c_{A,A} = c_{V,W} \circ (\alpha_v \otimes \alpha_w)$, from where:

$$c_{V,W}(v \otimes w) = (\alpha_w \otimes \alpha_v)(c_{A,A}(1 \otimes 1)) = \tau_{V,W}((\alpha_v \otimes \alpha_w)(R)) = \tau_{V,W}(R(v \otimes w)).$$

The A-linearity of $c_{A,A}$ means that $c_{A,A}(a \cdot (1 \otimes 1)) = a \cdot c_{A,A}(1 \otimes 1)$ for all $a \in A$, from where, using the previous relation, $\Delta(a)\tau_{A,A}(R) = \tau_{A,A}(R\Delta(a))$ or $\Delta^{op}(a)R = R\Delta(a)$. The commutativity of the hexagons with $U = V = W = A, \alpha_{A\otimes A} = \Delta$ implies the remaining relations for R. $\square$

Example 1.18 *Sweedler's 4-dimensional Hopf algebra.*

Let A be the algebra generated by two elements x and y and relations

$$x^2 = 1, \quad y^2 = 0, \quad yx + xy = 0.$$

The set $\{1, x, y, xy\}$ forms a basis of the vector space underlying A. There is a unique Hopf algebra structure on A such that

$$\Delta(x) = x \otimes x,\ \Delta(y) = 1 \otimes y + y \otimes x,\ S(x) = x,\ S(y) = xy,\ \varepsilon(x) = 1,\ \varepsilon(y) = 0.$$

Observe that S is of order 4 and that, for any $a \in A$, we have $S^2(a) = xax^{-1}$. Let us put

$$R_q = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \frac{q}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y),$$

where $q \in k$. It is easy to show that R_q is a unversal R-matrix for A, so we have a family of concrete examples of braided Hopf algebras parameterized by q. Observe that $R_q^{-1} = \tau_{A,A}(R_q)$.

Definition 1.19 *A tensor functor $(F, \varphi_0, \varphi_2)$ between braided tensor categories $\mathcal{C}$ and $\mathcal{D}$ is said to be braided if, for any pair (V, W) of objects of $\mathcal{C}$, the square*

$$\begin{array}{ccc} F(V) \otimes F(W) & \xrightarrow{\varphi_2} & F(V \otimes W) \\ \Big\downarrow{\scriptstyle c_{F(V),F(W)}} & & \Big\downarrow{\scriptstyle F(c_{V,W})} \\ F(W) \otimes F(V) & \xrightarrow{\varphi_2} & F(W \otimes V) \end{array}$$

commutes. Let us mention important special class of braided categories

Definition 1.20 *A braided tensor category is said to be* **symmetric** *if its braiding verifies $c_{W,V} \circ c_{V,W} = id_{V \otimes W}$ for all objects V, W of this category. Such a braiding is called a* **symmetry**.

Note that for symmetric tensor categories the hexagon axioms are equivalent.

Example 1.21 *1. $Vec(k)$ or $Vec_f(k)$ with the usual flip.*

2. Let (A, Δ, ε) be a **cocommutative** *bialgebra: $\Delta = \tau_{A,A} \circ \Delta = \Delta^{op}$ with the flip $\tau_{A,A} : A \otimes A \to A \otimes A$. Then the usual flip $\tau_{V \otimes W} : V \otimes W \to W \otimes V$ is a symmetry in $Rep(A)$ - the universal R-matrix in this case is just $1 \otimes 1$.*

2 Lecture 2. The center of a tensor category. Quantum double of a finite group.

1) The center of a strict tensor category.

Now we give a construction which assigns to any strict tensor category $(\mathcal{C}, \otimes, I)$ a braided tensor category $\mathcal{Z}(\mathcal{C})$ called the **center** of $\mathcal{C}$.

Definition 2.1 *Objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(V, c_{-,V})$, where V is an object of $\mathcal{C}$ such that there exists $c_{-,V}$, a family of natural isomorphisms $c_{X,V} : X \otimes V \to V \otimes X$ defined for all objects X of $\mathcal{C}$, such that*

$$c_{X \otimes Y,V} = (c_{X,V} \otimes id_Y)(id_X \otimes c_{Y,V}) \quad \textit{for all} \quad X, Y \in Ob(\mathcal{C}). \tag{1}$$

A morphism from $(V, c_{-,V})$ to $(W, c_{-,W})$ is a morphism $f : V \to W$ in $\mathcal{C}$ such that

$$(f \otimes id_X)c_{X,V} = c_{X,W}(id_X \otimes f) \quad \textit{for all} \quad X \in Ob(\mathcal{C}). \tag{2}$$

Clearly, $(I, id_X) \in Ob(\mathcal{Z}(\mathcal{C}))$ and if $(V, c_{-,V}) \in Ob(\mathcal{Z}(\mathcal{C}))$, then $id_V : (V, c_{-,V}) \to (V, c_{-,V})$ is a morphism in $\mathcal{Z}(\mathcal{C})$; if f, g are composable morphisms in $\mathcal{Z}(\mathcal{C})$, then $g \circ f$ in $\mathcal{C}$ is a morphism in $\mathcal{Z}(\mathcal{C})$. So, the identity of $(V, c_{-,V})$ in $\mathcal{Z}(\mathcal{C})$ is id_V.

The naturality in Definition 2.1 means that the square

$$\begin{array}{ccc} X \otimes V & \xrightarrow{\quad c_{X,V} \quad} & V \otimes X \\ {\scriptstyle f \otimes id_V} \downarrow & & \downarrow {\scriptstyle id_V \otimes f} \\ Y \otimes V & \xrightarrow{\quad c_{Y,V} \quad} & V \otimes Y \end{array}$$

commutes for any morphism $f : X \to Y$ in $\mathcal{C}$.

Theorem 2.2 *The center $\mathcal{Z}(\mathcal{C})$ of a strict tensor category $(\mathcal{C}, \otimes, I)$ is a strict braided tensor category, where:*

(i) the tensor product $(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V\otimes W})$, where the morphism $c_{X,V\otimes W} : X \otimes V \otimes W \to V \otimes W \otimes X$ of $\mathcal{C}$ is defined, $\forall X \in Ob(\mathcal{C})$, by

$$c_{X,V\otimes W} = (id_V \otimes c_{X,W})(c_{X,V} \otimes id_W), \tag{3}$$

(ii) the unit object is (I, id_X);

(iii) the braiding is given by

$$c_{V,W} : (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V}).$$

Proof. (a) Given $(V, c_{-,V}), (W, c_{-,W}) \in Ob(\mathcal{Z}(\mathcal{C}))$, we show that so is $(V \otimes W, c_{-,V\otimes W})$. Indeed, by definition of $(V, c_{-,V}), (W, c_{-,W})$, $c_{X,V\otimes W}$ is an isomorphism of $\mathcal{C}$ natural in X. For all $X, Y \in Ob(\mathcal{C})$ we have:

$$c_{X\otimes Y,V\otimes W} = (id_V \otimes c_{X\otimes Y,W})(c_{X\otimes Y,V} \otimes id_W) =$$
$$= (id_V \otimes c_{X,W} \otimes id_Y)(id_{V\otimes X} \otimes c_{Y,W})\times$$
$$\times(c_{X,V} \otimes id_{Y\otimes W})(id_X \otimes c_{Y,V} \otimes id_W) =$$
$$= (id_V \otimes c_{X,W} \otimes id_Y)(c_{X,V} \otimes id_{W\otimes Y})\times$$
$$\times(id_{X\otimes V} \otimes c_{Y,W})(id_X \otimes c_{Y,V} \otimes id_W) =$$
$$= (c_{X,V\otimes W} \otimes id_Y)(id_X \otimes c_{Y,V\otimes W}).$$

Here the first and forth equalities follow from (3), the second one from (1), and the third one by the naturality of $\otimes$.

(b) Given $f : (V, c_{-,V}) \to (W, c_{-,W})$ and $f' : (V', c_{-,V'}) \to (W', c_{-,W'})$ morphisms of $\mathcal{Z}(\mathcal{C})$, we show that so is $f \otimes f'$. We have:

$$(f \otimes f' \otimes id_X)c_{X,V\otimes V'} =$$
$$= (f \otimes id_{W'} \otimes id_X)(id_V \otimes f' \otimes id_X)(id_V \otimes c_{X,V'})(c_{X,V} \otimes id_{V'}) =$$
$$= (f \otimes id_{W'} \otimes id_X)(id_V \otimes c_{X,W'})(id_V \otimes id_X \otimes f')(c_{X,V} \otimes id_{V'}) =$$
$$= (id_W \otimes c_{X,W'})(f \otimes id_X \otimes id_{W'})(c_{X,V} \otimes id_{W'})(id_X \otimes id_V \otimes f') =$$
$$= (id_W \otimes c_{X,W'})(c_{X,V} \otimes id_{W'})(id_X \otimes f \otimes id_{W'})(id_X \otimes id_V \otimes f') =$$
$$= c_{X,W\otimes W'}(id_X \otimes f \otimes f').$$

Here the first and forth equalities follow from (3) and from the naturality of $\otimes$, the second and forth ones from (2), and the third one from the definition of the tensor product of morphisms in $\mathcal{C}$.

Now it is clear that $\mathcal{Z}(\mathcal{C})$ is a strict tensor category because $\otimes$ is well defined on its objects and morphisms and has all needed properties because it does so in $\mathcal{C}$. Let us show that $\mathcal{Z}(\mathcal{C})$ is braided.

(c) $c_{V,W}$ is a morphism in $\mathcal{Z}(\mathcal{C})$ because, for all $X \in Ob(\mathcal{C})$, we have:

$$(c_{V,W} \otimes id_X)c_{X,V\otimes W} = (c_{V,W} \otimes id_X)(id_V \otimes c_{X,W})(c_{X,V} \otimes id_W) =$$

$$= c_{V\otimes X,W}(c_{X,V} \otimes id_W) = (id_W \otimes c_{X,V})c_{X\otimes V,W} =$$

$$= (id_W \otimes c_{X,V})(c_{X,W} \otimes id_V)(id_X \otimes c_{V,W}) = c_{X,W\otimes V}(id_X \otimes c_{V,W}).$$

Here the first and the last equalities follow from (3), the second and forth ones from (1), and the third one from the naturality of $c_{-,V}$.

(d) The morphism $c_{V,W}$ is invertible by definition and is natural with respect to morphisms of $\mathcal{C}$, hence to those of $\mathcal{Z}(\mathcal{C})$. Now the axioms of braiding in strict tensor categories follow from the definitions of $c_{-,V}$ and $c_{X,V\otimes W}$. $\square$

Remark 2.3 *For any strict braided tensor category* $(\mathcal{C}, \otimes, c)$*, the map* $V \to (V, c_{-,V})$ *can be extended to a strict braided tensor functor* $Z : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ *such that* $\Pi \circ Z = id_{\mathcal{C}}$*, where* $\Pi : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ *is the forgetful strict tensor fuctor:* $\Pi(V, c_{-,V}) = V$ *- see [1].*

2) Quantum double of a finite group.

Given a Hopf algebra, the quantum double construction, due to V.G. Drinfeld, allows to get a braided Hopf algebra. Here we consider the case of the Hopf algebra associated with a finite group algebra.

Definition 2.4 *a) A* **left action** *of a bialgebra* (A, Δ, ε) *on a unital algebra* M *is a linear map* $A \otimes M \to M$*,* $a \otimes m \mapsto a \cdot m$ *such that:*

$$a \cdot (xy) = (a_{(1)} \cdot x)(a_{(2)} \cdot y), \;\; a \cdot 1 = \varepsilon(a)1 \quad (a \in A, \;\; x, y \in M),$$

where $\Delta(a) := a_{(1)} \otimes a_{(2)}$ *is the Sweedler's* **leg notation***. If* $(A, \Delta, S, \varepsilon, *)$ *is a* $*$*-Hopf algebra and* M *is a* $*$*-algebra over* $\mathbb{C}$*, then we also require that*

$$(a \cdot x)^* = S(a)^* \cdot x^*.$$

b) **Crossed product** *of* A *by* M*:* $M \rtimes A = M \otimes A$ *as vector space equipped with the product*

$$[m \otimes a][n \otimes b] = [m(a_{(1)} \cdot n) \otimes a_{(2)}b],$$

In $*$*-case we also have* $[m \otimes a]^* = [a^*_{(1)} \cdot m^* \otimes a^*_{(2)}]$.

If $A = kG$, we have: $g \cdot (xy) = (g \cdot x)(g \cdot y), \; g \cdot 1 = 1$ for all $g \in G$, $[m \otimes g][n \otimes h] = [m(g \cdot n) \otimes gh]$.

Exercise. Check that the product in $M \rtimes A$ is associative with unit $1_M \otimes 1_A$.

The group algebra kG of a finite group G is a Hopf algebra with coproduct, antipode and counit:

$$\Delta(g) = g \otimes g,\ S(g) = g^{-1},\ \varepsilon(g) = 1 \quad (g \in G).$$

Its dual $Fun(G)$ is a Hopf algebra with coproduct, antipode and counit:

$$\Delta(e_g) = \sum_{uv=g} (e_u \otimes e_v),\ S(e_g) = e_{g^{-1}},\ \varepsilon(e_g) = \delta_{g,e},$$

where e_g is a characteristic function of the set $\{g\}$, $\delta_{g,1}$ is the Kronecker symbol, and 1 is the unit of G. We consider the action of kG on $Fun(G)$ by conjugation: $g{\cdot}e_h := e_{ghg^{-1}}$ and equip the vector space $D(G) = Fun(G) \otimes kG$ with the crossed product

$$(e_g \otimes 1)(1 \otimes h) = (e_g \otimes h),\ (1 \otimes h)(e_g \otimes 1) = e_{hgh^{-1}} \otimes h,$$

$(e_g \otimes h)_{g,h\in G}$ is a basis in $D(G)$. In order to get a braided Hopf algebra structure on $D(G)$, we define also the coproduct, counit, antipode and the universal R-matrix:

$$\Delta(e_g \otimes h) = \sum_{uv=g} (e_v \otimes h \otimes e_u \otimes h),\ \varepsilon(e_g \otimes h) = \delta_{g,e},$$

$$S(e_g \otimes 1) = e_{g^{-1}} \otimes 1,\ S(1 \otimes h) = 1 \otimes h^{-1},\ R = \sum_{g\in G} (1 \otimes g \otimes e_g \otimes 1).$$

Exercise. Check that $D(G)$ is indeed a braided Hopf algebra and that $S^2 = id$.

Theorem 2.5 *The braided tensor categories $\mathcal{Z}(Rep(G))$ and $Rep(D(G))$ are equivalent.*

We start the proof with the following

Lemma 2.6 *Let (A, Δ, ε) be a bialgebra, $V, c_{-,V}$ be an object of $\mathcal{Z}(Rep(A))$ and $\Delta_V : V \to V \otimes A$ be the map defined, for all $v \in V$, by $\Delta_V(v) = c_{A,V}(1 \otimes v)$. Then:*
(i) $(\Delta_V \otimes id)\Delta_V = (id \otimes \Delta)\Delta_V$;
(ii) $(id \otimes \varepsilon)\Delta_V = id_V$.
(iii) $\Delta(a)\Delta_V(v) = \sum_{(a)} \Delta_V(a_{(2)}v)(1 \otimes a_{(1)})$.
Conditions (i),(ii) mean that V is a right (A, Δ, ε)-comodule.

Proof. By convention, $\Delta_V(v) = \sum_{(v)}(v_V \otimes v_A) \in V \otimes A$ for any $v \in V$. The naturality of $c_{-,V}$ allows to express $c_{X,V}$ in terms of Δ_V for any A-module X. Indeed, given $x \in X$ and $\alpha_x : A \to X$ the unique A-linear map such that $\alpha_x : 1 \to x$, we have $(id_V \otimes \alpha_x)c_{A,V} = c_{X,V}(\alpha_x \otimes id_V)$, from where

$$c_{X,V}(x \otimes v) = \Delta_V(v)(1 \otimes x) = \sum_{(v)} (v_V \otimes v_A x). \tag{4}$$

Let us show (i). By (1) we have:

$$c_{X\otimes Y,V}(x \otimes y \otimes v) = \sum_{(v)} (v_V \otimes (v_A)_{(1)} x \otimes (v_A)_{(2)} y) =$$

$$(c_{X,V} \otimes id_Y)((id_X \otimes c_{Y,V})(x \otimes y \otimes v)) = \sum_{(v)} ((v_V)_V \otimes (v_V)_A x \otimes v_A y).$$

Setting $X = Y = A$ and $x = y = 1$, we get

$$\sum_{(v)} (v_V \otimes (v_A)_{(1)} \otimes (v_A)_{(2)} = \sum_{(v)} ((v_V)_V \otimes (v_V)_A \otimes v_A),$$

which proves (i).

We also have $c_{k,V} = id_V$ because $k = I$ is the unit object (this follows from (1)). This implies $c_{k,V}(1 \otimes v) = \sum_{(v)} \varepsilon(v_A) v_V = v$ which proves (ii).

Since $c_{X,V}$ is A-linear, then we have $a \cdot c_{X,V}(x \otimes v) = c_{X,V}(a \cdot (x \otimes v))$, for all $a \in A, v \in V, x \in X$, or

$$\Delta(a)\Delta_V(v)(1 \otimes x) = (\sum_{(a)} \Delta_V(a_{(2)}v)(1 \otimes a_{(1)}))(1 \otimes x).$$

Setting $X = A, x = 1$, we obtain (iii). In particular, if $A = kG$, $\Delta_V(h \cdot v)(1 \otimes h) = \Delta(h)\Delta_V(v)$, for all $h \in G, v \in V$. □

Corollary 2.7 *If $A = kG$, any V as above is a left $D(G)$-module.*

Proof. Taking in mind the crossed product structure of $D(G)$, it suffices to show that V is both kG- and $Fun(G)$-module and these actions verify

$$h \cdot (e_g \cdot v) = e_{hgh^{-1}} \cdot (h \cdot v) \quad \text{for all} \quad v \in V, g, h \in G.$$

First, let us precise the action of $Fun(G)$ on V. Since any A-comodule is automatically an A^*-module, so one can put $f \cdot v := (id \otimes f)(\Delta_V(v))\ \forall f \in Fun(G)$.

Exercise. Check that this is indeed a left action.

Then, relations (iii) and $< e_g, a > = < e_{hgh^{-1}}, hah^{-1} >$ $(a \in kG)$ give:

$$h \cdot (e_g \cdot v) = \sum_{(v)} < e_g, v_A > (h \cdot v_V) = \sum_{(v)} < e_{hgh^{-1}}, hv_A h^{-1} > (h \cdot v_V) =$$

$$\sum_{(v)} < e_{hgh^{-1}}, (h \cdot v)_A > (h \cdot v)_V = e_{hgh^{-1}} \cdot (h \cdot v).$$

Lemma 2.8 *If $A = kG, X$ is an A-module, V as above, then $c_{X,V}(x \otimes v) = \tau_{X,V}(R(x \otimes v))$ for all $x \in X, v \in V$.*

Proof. Using (4) and the definition of the action of A^* on V, we have, using the decomposition $a = \sum_g < e_g, a > g$ for all $a \in A$:

$$c_{X,V}(x \otimes v) = \sum_{(v)} (v_V \otimes v_A x) = \sum_{(v),g} (< e_g, v_A > v_V \otimes g \cdot x) =$$

$= \sum_g ((e_g \cdot v) \otimes g \cdot x) = \tau_{X,V}(R(x \otimes v)).$ □

Proof of Theorem 2.5.

(i) Let us define a faithful functor $\mathcal{F} : \mathcal{Z}(Rep(G)) \to Rep(D(G))$. Corollary 2.7 shows that the map $\mathcal{F}(V, c_{-,V}) := V$ is well defined on objects. Recall that the action of $D(G)$ on V is defined by

$$(gf) \cdot v = \sum_{(v)} < f, v_A > g \cdot v_V \quad \text{for all} \quad g \in G, f \in Fun(G), v \in V. \qquad (5)$$

If $\varphi : V \to W$ is a morphism in $\mathcal{Z}(Rep(G))$, then it is, by definition, a morphism in $Rep(G)$, but also, due to (2) and the definition of Δ_V, a morphism of $A = kG$-comodules (i.e., $\Delta_W(\varphi(v)) = (\varphi \otimes id_A)\Delta_V(v))$, hence of $Fun(G)$-modules. Thus, φ is $D(G)$-linear and $\mathcal{F}$ is a faithful functor.

(ii) Let us show that $\mathcal{F}$ is a strict tensor functor. Recall that $(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V\otimes W})$, where $c_{-,V\otimes W}$ is determined by $c_{A,V\otimes W} = (id_V \otimes c_{A,W})(c_{A,V} \otimes id_W)$, therefore,

$$\Delta_{V\otimes W}(v \otimes w) = \sum_{(v),(w)} v_V \otimes w_W \otimes w_A v_A$$

- this is the tensor product of the right comodule structures on V and W, and from (5) we have (using the definition of Δ on $A^* = Fun(G)$):

$$f \cdot (v \otimes w) = \sum_{(v),(w)} < f, (w_A v_A) > v_V \otimes w_W =$$

$$= \sum_{(v),(w)} < \Delta(f), v_A \otimes w_A > v_V \otimes w_W = \Delta(f) \cdot (v \otimes w).$$

So, the action of $D(G)$ on $V \otimes W$, for all $g \in G, f \in Fun(G)$, is given by

$$(g \cdot f)(v \otimes w) = \Delta(g)[\Delta(f) \cdot (v \otimes w)] = \Delta(g \cdot f) \cdot (v \otimes w),$$

which is the action given by the coproduct of $D(G)$.

(iii) The tensor functor $\mathcal{F}$ is braided because, by definition of the braiding in $\mathcal{Z}(Rep(G))$, Lemma 2.8 gives $\mathcal{F}(c_{V,W})(v \otimes w) = \tau_{V,W}(R(v \otimes w))$, which is the braiding in $Rep(D(G))$.

(iv) Let us construct a functor $\mathcal{G} : Rep(D(G)) \to \mathcal{Z}(Rep(G))$. For any $D(G)$-module V and $A = kG$-module X, let us define $c_{X,V}$ by

$$c_{X,V}(x \otimes v) = \tau_{X,V}(R(x \otimes v)) \quad \text{for all} \quad x \in X, v \in V.$$

Let us show that $\mathcal{G}(V) = (V, c_{-,V})$ is an object of $\mathcal{Z}(Rep(G))$. Since R is invertible, $c_{X,V} : X \otimes V \to V \otimes X$ is an isomorphism. It is A-linear because, for all $a \in A = kG$:

$$c_{X,V}(a(x \otimes v)) = \tau_{X,V}(R\Delta(a)(x \otimes v)) = \tau_{X,V}(\Delta^{op}(a)R(x \otimes v)) =$$

$$= \Delta(a)\tau_{X,V}(R(x \otimes v)) = a \cdot c_{X,V}(x \otimes v).$$

We also have to check (1), i.e., the relation

$$c_{X\otimes Y,V}(x \otimes y \otimes v) = (c_{X,V} \otimes id_Y)(id_X \otimes c_{Y,V})(x \otimes y \otimes y \otimes v).$$

The left-hand side equals to $\tau_{X\otimes Y,V}((\Delta \otimes id_A)(R)(x \otimes y \otimes v)$ and the right-hand side equals to $\tau_{X\otimes Y,V}(R_{12}R_{13}(x \otimes y \otimes v)$, so the above equality holds by the definition of R. This means that $\mathcal{G}(V) = (V, c_{-,V})$ is an object of $\mathcal{Z}(Rep(G))$.

Let us check that $\mathcal{G}(f) := f$ (where $f : V \to W$ is a morphism in $Rep(D(G))$) is a morphism in $\mathcal{Z}(Rep(G))$. By definition, it is A-linear. Then,

$$((f \otimes id_X)c_{X,V})(x \otimes v) = \tau_{X,W}((id_X \otimes f)(R(x \otimes v))) =$$

$$= \tau_{X,W}(R(x \otimes f(v))) = c_{X,W}((id_X \otimes f)(x \otimes v))$$

for all $x \in X, v \in V$. This proves (2).

(v) Clearly, $\mathcal{F} \circ \mathcal{G} = id$. Lemma 2.8 implies $\mathcal{G} \circ \mathcal{F} = id$, so the braided tensor categories $Rep(D(G))$ and $\mathcal{Z}(Rep(G))$ are equivalent. □

3 Lecture 3. Duality. Ribbon categories and ribbon Hopf algebras.

1) Duality.

Definition 3.1 *A strict tensor category $(\mathcal{C}, \otimes, I)$ is said to have a left duality if for each $V \in Ob(\mathcal{C})$ there exist $V^* \in Ob(\mathcal{C})$ and morphisms*

$$b_V : I \to V \otimes V^* \quad \textit{and} \quad d_V : V^* \otimes V \to I$$

in $\mathcal{C}$ such that

$$(id_V \otimes d_V)(b_V \otimes id_V) = id_V \quad \textit{and} \quad (d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*} \qquad (6)$$

Example 3.2 *1. Let us consider the strict tensor category $Vec_f(k)$, let V be an object of this category and V^* be its dual vector space. Let us define the maps $b_V : k \to V \otimes V^*$ and $d_V : V^* \otimes V \to k$ by*

$$b_V(1) = \sum_i v_i \otimes v^i \quad \textit{and} \quad d_V(v^i \otimes v_j) =< v^i, v_j >,$$

where $\{v_i\}_i$ is any basis of V and $\{v^i\}_i$ is the dual basis of V^*.

Exercise. *Check that these definitions do not depend on the choice of the bases and that these maps verify the conditions (6).*

2. Let $(A, \Delta, S, \varepsilon)$ be a Hopf algebra. Consider the strict tensor category $Rep_f(A)$ of finite-dimensional left A-modules which is a tensor subcategory of $Rep(A)$. Given an object V of $Rep_f(A)$, we can equip the dual vector space $V^* = Hom(V, k)$ with the left action of A given by

$$< a \cdot f, v >:=< f, S(a) \cdot v > \quad \textit{for all} \quad a \in A, v \in V, f \in V^*.$$

Let us define, as above, the maps $b_V : k \to V \otimes V^*$ and $d_V : V^* \otimes V \to k$ by

$$b_V(1) = \sum_i v_i \otimes v^i \quad \textit{and} \quad d_V(v^i \otimes v_j) =< v^i, v_j >,$$

where $\{v_i\}_i$ is any basis of V and $\{v^i\}_i$ is the dual basis in V^*. Let us show that they are A-linear. For all $a \in A, v \in V, f \in V^*$ we have:

$$d_V(a \cdot (f \otimes v)) = d_V(a_{(1)} \cdot f) \otimes (a_{(2)} \cdot v)) =< (a_{(1)} \cdot f), (a_{(2)} \cdot v) >=$$

$$=< f, S(a_{(1)})a_{(2)} \cdot v >=< f, \varepsilon(a)v >= \varepsilon(a)d_V(f \otimes v) = a \cdot d_V(f \otimes v),$$

$$a \cdot b_V(1) = \sum_i (a_{(1)} \cdot v_i) \otimes (a_{(2)} \cdot v^i) = \sum_{i,j} (a_{(1)} \cdot v_i) \otimes < a_{(2)} \cdot v^i, v_j > v^j =$$

$$= \sum_j (a_{(1)} \sum_i < v^i, S(a_{(2)}) \cdot v_j > \cdot v_i) \otimes v^j = \sum_j (a_{(1)} S(a_{(2)}) \cdot v_j) \otimes v^j =$$

$$= \varepsilon(a) \sum_j v_j \otimes v^j = b_V(a \cdot 1).$$

Now let us show that d_V and b_V equip the tensor category $Rep_f(A)$ with a left duality. We compute:

$$(id_V \otimes d_V)(b_V \otimes id_V)(v) = (id_V \otimes d_V)(b_V(1) \otimes v) = (id_V \otimes d_V) \sum_i (v_i \otimes v^i \otimes v) =$$

$$= \sum_i < v^i, v > v_i = v,$$

$$(d_V \otimes id_{V^*})(id_{V^*} \otimes b_V)(f) = (d_V \otimes id_{V^*})(f \otimes b_V(1)) = (d_V \otimes id_{V^*}) \sum_i (f \otimes v_i \otimes v^i) =$$

$$= \sum_i < f, v_i > v^i = f.$$

Lemma 3.3 *Given $V \in Ob(\mathcal{C})$, V^* is unique up to a unique isomorphism compatible with d_V and b_V, i.e., for any two duals, $(V^*_{(1)}, d^{(1)}_V, b^{(1)}_V)$ and $(V^*_{(2)}, d^{(2)}_V, b^{(2)}_V)$ of V, there is a unique isomorphism $\varphi : V^*_{(1)} \to V^*_{(2)}$ such that $d^{(1)}_V = d^{(2)}_V(\varphi \otimes id_V)$, $b^{(2)}_V = (id_V \otimes \varphi)b^{(1)}_V$.*

Proof. Put $\varphi = (d^{(1)}_V \otimes id_{V^*_{(2)}})(id_{V^*_{(1)}} \otimes b^{(2)}_V)$, then

$$(id_V \otimes \varphi)b^{(1)}_V = (id_V \otimes d^{(1)}_V \otimes id_{V^*_{(2)}})(id_V \otimes id_{V^*_{(1)}} \otimes b^{(2)}_V)b^{(1)}_V =$$

$$= (id_V \otimes d^{(1)}_V \otimes id_{V^*_{(2)}})(b^{(1)}_V \otimes id_V \otimes id_{V^*_{(2)}})b^{(2)}_V = b^{(2)}_V$$

and similarly one can prove the other relation. φ is an isomorphism because, putting $\varphi^{-1} = (d^{(2)}_V \otimes id_{V^*_{(1)}})(id_{V^*_{(2)}} \otimes b^{(1)}_V) : V^*_{(2)} \to V^*_{(1)}$, we have, for example:

$$\varphi^{-1} \circ \varphi = (d^{(2)}_V \otimes id_{V^*_{(1)}})(id_{V^*_{(2)}} \otimes b^{(1)}_V)\varphi =$$

$$= (d^{(2)}_V \otimes id_{V^*_{(1)}})(\varphi \otimes id_V \otimes id_{V^*_{(1)}})(id_{V^*_{(1)}} \otimes b^{(1)}_V) =$$

$$= (d^{(1)}_V \otimes id_{V^*_{(1)}})(id_{V^*_{(1)}} \otimes b^{(1)}_V) = id_{V^*_{(1)}}$$

and similarly one proves that $\varphi \circ \varphi^{-1} = id_{V^*_{(2)}}$. □

Let us define $f^* : V^* \to U^*$ for a morphism $f : U \to V$ in $\mathcal{C}$ by

$$f^* = (d_V \otimes id_{U^*})(id_{V^*} \otimes f \otimes id_{U^*})(id_{V^*} \otimes b_U).$$

This allows to extend duality to a functor $\mathcal{C} \to \mathcal{C}$. Indeed, we have

Proposition 3.4 *Let $\mathcal{C}$ be a strict tensor category with left duality.*

(a) If $f : V \to W,\ g : U \to V$ are two morphisms, then $(f \circ g)^* = g^* \circ f^*$ and $(id_V)^* = id_{V^*}$.

(b) For any $U, V, W \in Ob(\mathcal{C})$, we have natural bijections:

$$Hom(U \otimes V, W) \cong Hom(U, W \otimes V^*), \quad \text{and} \quad Hom(U^* \otimes V, W) \cong Hom(V, U \otimes W).$$

(c) For any pair (V, W) of objects of $\mathcal{C}$, the objects $(V \otimes W)^*$ and $W^* \otimes V^*$ are isomorphic.

Proof. (a) **Exercise.** Check that $(id_V)^* = id_{V^*}$.

Now, for $f : V \to W,\ g : U \to V$ we have:

$$g^* \circ f^* = (d_V \otimes id_{U^*})(id_{V^*} \otimes g \otimes id_{U^*})(id_{V^*} \otimes b_U) \circ f^* =$$

$$= (d_V(f^* \otimes g) \otimes id_{U^*})(id_{W^*} \otimes b_U) =$$

$$= (d_V[(d_W \otimes id_{V^*})(id_{W^*} \otimes f \otimes id_{V^*})(id_{W^*} \otimes b_V) \otimes g] \otimes id_{U^*})(id_{W^*} \otimes b_U) =$$

$$= (d_W \otimes id_{U^*})(id_{W^*} \otimes (f \circ (id_V \otimes d_V)(b_V \otimes id_V) \circ g) \otimes id_{U^*})(id_{W^*} \otimes b_U) = (f \circ g)^*.$$

(b) For $f \in Hom(U \otimes V, W)$ and $g \in Hom(U, W \otimes V^*)$, we define elements

$$f^\sharp = (f \otimes id_{V^*})(id_U \otimes b_V) \quad \text{and} \quad g^\flat = (id_W \otimes d_V)(g \otimes id_V)$$

of $Hom(U, W \otimes V^*)$ and $Hom(U \otimes V, W)$, respectively. The definition of duality implies that $(f^\sharp)^\flat = f$ and $(g^\flat)^\sharp = g$. Indeed,

$$(f^\sharp)^\flat = (id_W \otimes d_V)(f \otimes id_{V^*} \otimes id_V)(id_U \otimes b_V \otimes id_V) =$$

$$= f \circ (id_U \otimes id_V \otimes d_V)(id_U \otimes b_V \otimes id_V) = f \circ (id_U \otimes id_V) = f,$$

$$(g^\flat)^\sharp = (id_W \otimes d_V \otimes id_{V^*})(g \otimes id_V \otimes id_{V^*})(id_U \otimes b_V) =$$

$$(id_W \otimes d_V \otimes id_{V^*})(id_W \otimes id_{V^*} \otimes b_V) \circ g = g.$$

The other bijection can be proved similarly.

c) Due to Lemma 3.3, it suffices to show that $W^* \otimes V^*$ is dual to $V \otimes W$ with $d_{V \otimes W} = d_W(id_{W^*} \otimes d_V \otimes id_W)$ and $b_{V \otimes W} = (id_V \otimes b_W \otimes id_V)b_V$. For example, we have:

$$(id_{V \otimes W} \otimes d_{V \otimes W})(b_{V \otimes W} \otimes id_{V \otimes W}) = (id_{V \otimes W} \otimes d_W)(id_{V \otimes W} \otimes id_{W^*} \otimes d_V \otimes id_W) \times$$

$$\times (id_V \otimes b_W \otimes id_{V^*} \otimes id_{V \otimes W})(b_V \otimes id_{V \otimes W}) = (id_{V \otimes W} \otimes d_W)(id_V \otimes b_W \otimes d_V \otimes id_W) \times$$

$$\times (b_V \otimes id_{V \otimes W}) = (id_V \otimes id_W \otimes d_W)(id_V \otimes b_W \otimes id_W) = id_{V \otimes W}$$

and similarly one proves that $(d_{V \otimes W} \otimes id_{W^* \otimes V^*})(id_{W^* \otimes V^*} \otimes b_{V \otimes W}) = id_{W^* \otimes V^*}$. $\square$

Remark 3.5 *Explicitly, if we define morphisms $\lambda_{V,W} : W^* \otimes V^* \to (V \otimes W)^*$ and $\lambda^{-1}_{V,W} : (V \otimes W)^* \to W^* \otimes V^*$, respectively, by*

$$\lambda_{V,W} = (d_W \otimes id_{(V \otimes W)^*})(id_{W^*} \otimes d_V \otimes id_{W \otimes (V \otimes W)^*})(id_{W^* \otimes V^*} \otimes b_{V \otimes W}),$$

$$\lambda^{-1}_{V,W} = (d_{V \otimes W} \otimes id_{W^* \otimes V^*})(id_{(V \otimes W)^* \otimes V} \otimes b_W \otimes id_{V^*})(id_{(V \otimes W)^*} \otimes b_V),$$

then one can check that $\lambda^{-1}_{V,W}$ is indeed inverse to $\lambda_{V,W}$.

There is a similar notion of a right duality: we say that a strict tensor category $(\mathcal{C}, \otimes, I)$ has a right duality if for each object V of $\mathcal{C}$ there exist an object ${}^*V$ and morphisms

$$b'_V : I \to^* V \otimes V \quad \text{and} \quad d'_V : V \otimes^* V \to I$$

of this category such that

$$(d'_V \otimes id_V)(id_V \otimes b'_V) = id_V \quad \text{and} \quad (id_{^*V} \otimes d'_V)(b'_V \otimes id_{^*V}) = id_{^*V}.$$

Then we define, for any morphism $f : V \to W$, a morphism ${}^*f :^W \to^* V$ by

$${}^*f = (id_{^*V} \otimes d'_W)(id_{^*V} \otimes f \otimes id_{^*W})(b'_V \otimes id_{^*W})$$

and prove, like in the previous proposition, that the map $V \to^* V$ can be extended to a functor.

In general, left and right dualities are different, but if $\mathcal{C}$ has right and left duality (such categories are called autonomous), then one can show that ${}^*(V^*) \cong V \cong ({}^*V)^*$ for any object V. The proof is based on the following natural isomorphisms:

$$Hom(U,^* (V)^* \otimes W) \cong Hom(V^* \otimes U, W) \cong Hom(U, V \otimes W),$$

the first one being implied by the right, and the second one - by the left duality.

Example 3.6 *1. The right duality in the category $Vec_f(k)$ can be defined, for any object V and its dual ${}^*V = V^*$, by the maps*

$$b'_V(1) = \sum_i v^i \otimes v_i \quad \textit{and} \quad d'_V(v_i \otimes v^j) =< v^j, v_i >$$

using the same notations as above. So, the category $Vec_f(k)$ is autonomous.

If the antipode S of a Hopf algebra $(A, \Delta, S, \varepsilon)$ is invertible and V is an object of $Rep_f(A)$, we can equip the same dual vector space ${}^*V = V^* = Hom(V, k)$ with another left action of A given by

$$< a \cdot f, v >:=< f, S^{-1}(a) \cdot v > \quad \textit{for all} \quad a \in A, v \in V, f \in^* V$$

and introduce maps $b'_V : k \to^* V \otimes V$ and $d'_V : V \otimes^* V \to k$ by

$$b'_V(1) = \sum_i v^i \otimes v_i \quad \textit{and} \quad d'_V(v_i \otimes v^j) =< v^j, v_i >$$

using the same notations as above. Then one can check that these maps are A-linear and equip the strict tensor category $Rep_f(A)$ with a right duality, so this category is autonomous.

2) Ribbon categories.

Definition 3.7 *A strict braided tensor category $(\mathcal{C}, \otimes, I, c)$ with left duality is said to be* **ribbon** *if it has a family $\theta : V \to V$ of natural isomorphisms indexed by the objects V of $\mathcal{C}$ such that*

$$\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c_{W,V} \circ c_{V,W} \quad \textit{and} \quad \theta_{V^*} = \theta_V^*.$$

Such a family θ_V is called a **twist**. *Its naturality means that $\theta_W \circ f = f \circ \theta_V$ for any morphism $f : V \to W$.*

Lemma 3.8 *a)* $\theta_I = id_I$.

b)For all objects V, W *of a ribbon category* $\mathcal{C}$ *we have*

$$\theta_{V\otimes W} = c_{W,V} \circ c_{V,W}(\theta_V \otimes \theta_W) = c_{W,V}(\theta_W \otimes \theta_V)c_{V,W}.$$

Proof. *a) If* $V = W = I$, *the definition of a twist gives* $\theta_{I\otimes I} = (\theta_I \otimes \theta_I)c_{I,I}c_{I,I}$. *But the first hexagon axiom in the definition of braiding with* $U = V = W = I$ *implies for strict tensor categories:* $c_{I,I} = c_{I,I} \circ c_{I,I}$, *so that* $c_{I,I} = id_I$. *Now, the naturality of the identification of* $V \otimes I$ *with* I *gives* $\theta_{I\otimes I} = \theta_I \otimes id_I = id_I \otimes \theta_I$ *which gives the first statement.*

b) Follows from the naturality of $c_{V,W}$ *which gives* $(\theta_W \otimes \theta_V)c_{V,W} = c_{V,W}(\theta_V \otimes \theta_W)$ *for all* $V, W \in Ob(\mathcal{C})$. □

Example 3.9 *1.* Vec_f *is a ribbon category with the trivial twist* $\theta_V = id_V$.

2. **Exercise.** *Show that any symmetric tensor category* $\mathcal{C}$ *with left duality is a ribbon category with the trivial twist* $\theta_V = id_V$. *In particular, such is the category* $Rep_f(A)$, *where* A *is a cocommutative Hopf algebra or a braided Hopf algebra whose universal R-matrix* r *verifies* $\tau_{A,A}(R) = R^{-1}$.

Using the braiding and the twist, we can define morphisms $b'_V : I \to V^* \otimes V$ and $d'_V : V \otimes V^* \to I$ for any object V of a ribbon category $\mathcal{C}$ by

$$b'_V = (id_{V^*} \otimes \theta_V)c_{V,V^*} \circ b_V \quad \text{and} \quad d'_V = d_V \circ c_{V,V^*}(\theta_V \otimes id_{V^*}).$$

It can be shown (see [1]) that b'_V and d'_V equip $\mathcal{C}$ with right duality, where ${}^*V = V^*$ and that the object $V^{**} = (V^*)^*$ is canonically isomorphic to V for all $V, W \in Ob(\mathcal{C})$.

3) Ribbon Hopf algebras.

Let $(A, \Delta, S, \varepsilon, R)$ be a braided Hopf algebra with a universal R-matrix $R = R_{(1)} \otimes R_{(2)}, R^{-1} = (R^{-1})_{(1)} \otimes (R^{-1})_{(2)} \in A \otimes A$, and let us put $u = S(R_{(2)})R_{(1)}$. This element is called the **Drinfeld element** of a braided Hopf algebra.

Lemma 3.10 $u^{-1} = S^{-1}((R^{-1})_{(2)})(R^{-1})_{(1)}$, $uS(u) = S(u)u \in Z(A)$, $\Delta(u) = (R_{21}R)^{-1}(u \otimes u)$, $\varepsilon(u) = 1$ *and* $S^2(a) = uau^{-1}$ *for all* $a \in A$.

Proof. (a) First, we show that $S(a_{(2)})ua_{(1)} = \varepsilon(a)u$ for all $a \in A$. Indeed, using properties of R and the axioms of a Hopf algebra, we compute:

$$S(a_{(2)})ua_{(1)} = S(a_{(2)})S(R_{(2)})R_{(1)}a_{(1)} = S(R_{(2)}a_{(2)})R_{(1)}a_{(1)} =$$

$$= S(a_{(1)}R_{(2)})a_{(2)}R_{(1)} = S(R_{(2)})S(a_{(1)})a_{(2)}R_{(1)} = \varepsilon(a)u.$$

Using this relation and again the axioms of a Hopf algebra, we have, denoting $(id \otimes \Delta)\Delta(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$:

$$ua = S(\varepsilon(a_{(2)})1)ua_{(1)} = S(a_{(2)}S(a_{(3)}))ua_{(1)} = S^2(a_{(3)})S(a_{(2)})ua_{(1)} =$$

$$S^2(a_{(2)})\varepsilon(a_{(1)})u = S^2(a)u.$$

Using this relation, we can now check that u is invertible and $u^{-1} = v = S^{-1}((R^{-1})_{(2)})(R^{-1})_{(1)}$. Indeed,

$$uv = uS^{-1}((R^{-1})_{(2)})(R^{-1})_{(1)} = S((R^{-1})_{(2)})u(R^{-1})_{(1)} =$$

$$= S((R^{-1})_{(2)})S(R_{(2)})R_{(1)}(R^{-1})_{(1)} = S(R_{(2)}(R^{-1})_{(2)})R_{(1)}(R^{-1})_{(1)} = 1$$

and $1 = uv = S^2(v)u$. Thus, $S^2(a) = uau^{-1}$ for all $a \in A$, in particular, $S^2(u) = u$.

(b) Let us show that $uS(u) = S(u)u \in Z(A)$. Relation $ua = S^2(a)u$, for any $a \in A$, implies $S(a)S(u) = S(u)S^3(a)$ or, replacing a by $S^{-1}(a)$, $aS(u) = S(u)S^2(a) = S(u)uau^{-1}$. Therefore, $aS(u)u = S(u)ua$, so $S(u)u \in Z(A)$. Putting $a = u$, we have $uS(u) = S(u)u$.

(c) Using the axioms of a Hopf algebra we have $\varepsilon(u) = \varepsilon(S(R_{(2)})R_{(1)}) = \varepsilon(S(R_{(2)})\varepsilon(R_{(1)}) = \varepsilon(S(\varepsilon(R_{(1)})R_{(2)})) = 1$, the last equality due to the relation $(\varepsilon \otimes id_A)(R) = 1$ (see exercise in Lecture 1).

(d) Let us compute $\Delta(u)$. Applying the flip $\tau_{A,A}$ to the relation $\Delta^{op}(a)R = R\Delta(a)$, we get $\Delta(a)R_{21} = R_{21}\Delta^{op}(a)$, and using again the above mentioned relation, we get $\Delta(a)R_{21}R = R_{21}R\Delta(a)$ for all $a \in A$. So, to get the needed result for $\Delta(u)$, it suffices to show that $\Delta(u)R_{21}R = u \otimes u$. We compute, using the last relation:

$$\Delta(u)R_{21}R = \Delta(S(R_{(2)})R_{(1)})R_{21}R =$$

$$= (S \otimes S)\Delta^{op}(R_{(2)})\Delta(R_{(1)})R_{21}R = (S \otimes S)\Delta^{op}(R_{(2)})R_{21}R\Delta(R_{(1)}).$$

Now consider the following right action of the algebra $A \otimes A \otimes A \otimes A$ on $A \otimes A$:

$$(a \otimes b) \cdot (X \otimes Y) := (S \otimes S)(X)(a \otimes b)Y, \quad \text{where} \quad a, b \in A,\ X, Y \in A \otimes A.$$

Then the right hand side of the last equality can be viewed as the action on R_{21} of the element $R\Delta(R_{(1)}) \otimes \Delta^{op}(R_{(2)}) = (R \otimes 1 \otimes 1)(R_{(1)} \otimes 1 \otimes \Delta^{op}(R_{(2)}))(1 \otimes R_{(1)}\Delta^{op}(R_{(2)})) = R_{12}R_{13}R_{23}R_{14}R_{24} = R_{23}R_{13}R_{12}R_{14}R_{24}$, and we can evaluate this element step by step.

Using the formula $R^{-1} = (id_A \otimes S^{-1})(R)$ from Lecture 1, we get:

$$R_{21} \cdot R_{23} = (S \otimes S)(R'_{(2)} \otimes 1)R_{21}(1 \otimes R'_{(1)}) = S(R'_{(2)})R_{(2)} \otimes R_{(1)}R'_{(1)} =$$

$$= (S \otimes id_A)(S^{-1}(R_{(2)})R'_{(2)}) \otimes R_{(1)}R'_{(1)} = 1 \otimes 1.$$

Hence, $R_{21} \cdot (R_{23}R_{13}) = (1 \otimes 1) \cdot R_{13} = (S \otimes S)(R_{(2)} \otimes 1)(R_{(1)} \otimes 1) = u \otimes 1$. Next,

$$R_{21} \cdot (R_{23}R_{13}R_{12}) = (u \otimes 1) \cdot R_{12} = (u \otimes 1)R$$

and, using again the formula $R^{-1} = (id_A \otimes S^{-1})(R)$,

$$R_{21} \cdot (R_{23}R_{13}R_{12}R_{14}) = (u \otimes 1)R \cdot R_{12} = (S \otimes S)(1 \otimes R'_{(2)})(u \otimes 1)R(R'_{(1)} \otimes 1) =$$

$$= (u \otimes 1)(R_{(1)}R'_{(1)} \otimes S(S^{-1}(R_{(2)})R'_{(2)}) = u \otimes 1.$$

Finally, $$R_{21} \cdot (R_{23}R_{13}R_{12}R_{14}R_{24}) =$$

$$= (u \otimes 1) \cdot R_{24} = (S \otimes S)(1 \otimes R_{(2)})(u \otimes 1)(1 \otimes R_{(1)}) = (u \otimes u),$$

so we have the needed result. □

Definition 3.11 *A braided Hopf algebra* $(A, \Delta, S, \varepsilon, R)$ *is said to be a* **ribbon Hopf algebra** *if there exists an invertible element* $\theta \in Z(A)$ *such that*

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta.$$

Relation between ribbon categories and ribbon Hopf algebras is given by the following

Proposition 3.12 *For any ribbon Hopf algebra* A *with* $\theta \in Z(A)$ *as above, the strict tensor category* $Rep_f(A)$ *is ribbon with twist* θ_V *defined on any finite-dimensional* A*-module* V *by the action of* θ^{-1}.

Conversely, if A *is a finite-dimensional braided Hopf algebra and the braided category* $Rep_f(A)$ *with left duality is ribbon, then* A *is a ribbon Hopf algebra.*

Proof. (a) Let A be a ribbon Hopf algebra with the distinguished invertible element $\theta \in Z(A)$. Then we have explained above that $Rep_f(A)$ is a braided category with left and right duality. Let us define an endomorphism of any object V of this category by $\theta_V(v) := \theta^{-1} \cdot v$ for any $v \in V$. Since $\theta \in Z(A)$ and is invertible, θ_V is an A-linear endomorphism of V. Let us prove that it is a twist:

$$(\theta_V \otimes \theta_W)c_{W,V}c_{V,W}(v \otimes w) = (\theta^{-1} \otimes \theta^{-1})(R_{21}R)(v \otimes w) =$$

$$= \Delta(\theta^{-1})(v \otimes w) = \theta_{V \otimes W}(v \otimes w)$$

for all $v \in V, w \in W$ and, for all $v \in V, \alpha \in V^*$:

$$< (\theta_V)^*(\alpha), v >=< \alpha, \theta_V(v) >=< \alpha, \theta^{-1}(v) >=< \alpha, S(\theta^{-1})(v) >=$$

$$=< \theta^{-1}\alpha, v >=< \theta_{V^*}(\alpha), v > .$$

(b) We now assume that the Hopf algebra $(A, \Delta, S, \varepsilon)$ is finite-dimensional and that the category $Rep_f(A)$ is ribbon. In particular, $Rep_f(A)$ is braided which implies that A is braided. Since $\dim(A) < +\infty$, it can be viewed as an object of the category $Rep_f(A)$, so we can consider the corresponding twist θ_A. Let us define $\theta := (\theta_A(1))^{-1}$. By the naturality of the twist, we have for any object V of $Rep_f(A)$ and for any $v \in V : \ \theta_V(v) = \theta_A(1)v = \theta^{-1}v$. The A-linearity of θ_A implies that $\theta \in Z(A)$. The relations in the definition of a twist imply, respectively,

$$\Delta(\theta^{-1}) = (\theta^{-1} \otimes \theta^{-1})(R_{21}R), \quad \text{and} \quad S(\theta^{-1}) = \theta^{-1}.$$

Finally, the relation $\varepsilon(\theta) = 1$ follows from Lemma 3.8 (a). □

One can show (see [1]) that this proposition implies the following

Corollary 3.13 *The element* θ^2 *of a ribbon Hopf algebra* A *acts as* $uS(u)$ *on any* $V \in Rep_f(A)$*, so* $\theta^2 = uS(u)$.

4) Quantum trace and quantum dimension in ribbon categories.

Applications of ribbon categories and ribbon Hopf algebras to computation of invariants of knots and 3-dimentional varieties (see [2]) are heavily based on the notions of quantum trace of endomorphisms and of quantum dimension of objects of a ribbon category.

Definition 3.14 *For any object V of a ribbon category $\mathcal{C}$ and any endomorphism f of V, the* **quantum trace** $tr_q(f)$ *of f is defined as the following element of the monoid $End(I)$:*

$$tr_q(f) = d'_V(f \otimes id_{V^*})b_V = d_V c_{V,V^*}((\theta_V \circ f) \otimes id_{V^*})b_V.$$

Exercise. Show that this definition gives the usual trace if $\mathcal{C} = Vec_f(k)$.

We formulate without proof the following

Theorem 3.15 *If f and g are endomorphisms in a ribbon category, then:*

(a) $tr_q(f \circ g) = tr_q(g \circ f)$ *whenever f and g are composable.*

(b) $tr_q(f \otimes g) = tr_q(f)tr_q(g)$, *and*

(c) $tr_q(f) = tr_q(f^*)$ *in the monoid $End(I)$.*

Definition 3.16 *For any object V of a ribbon category, the quantum dimension is defined by*

$$dim_q(V) = tr_q(id_V) = d'_V \circ b_V \in End(I).$$

Corollary 3.17 *For any objects V and W of a ribbon category we have*

$$dim_q(V \otimes W) = dim_q(V) \circ dim_q(W), \quad dim_q(V^*) = dim_q(V).$$

Now we are able to compute quantum trace and quantum dimension in the category $Rep_f(A)$ over a ribbon Hopf algebra A.

Proposition 3.18 *Let $f \in End(V), V \in Ob(Rep_f(A))$, where A is a ribbon Hopf algebra. Then*

$$tr_q(f) = tr(v \mapsto \theta^{-1} u f(v)).$$

In particular, $\dim_q(V)$ *equals to the trace of the linear map $v \mapsto \theta^{-1}u \cdot v$ on V.*

Proof. Using the definitions of d'_V and of u and the Proposition 3.12, we get:

$d'_V(v \otimes \alpha) =< R_{(2)} \cdot \alpha, R_{(1)}\theta^{-1} \cdot v >=< \alpha, S(R_{(2)})R_{(1)}\theta^{-1} \cdot v >=< \alpha, u\theta^{-1} \cdot v >$,

therefore,

$$tr_q(f) = d'_V(f \otimes id_{V^*})b_V = \sum_i < v^i, \theta^{-1}u \cdot f(v_i) >,$$

which is the usual trace of the linear endomorphism $v \mapsto \theta^{-1}u \cdot f(v)$. □

Example 3.19 *(Sweedler's 4-dimensional Hopf algebra).*

Let us consider the braided Hopf algebra of Example 1.18 and compute that $u = S(u) = x$ independently on q. This gives $uS(u) = x^2 = 1$, so this Hopf algebra is ribbon with $\theta = 1$.

References

[1] Ch. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Springer-Verlag, **155** (1995), 551pp.

[2] Ch. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et Synthèses, Soc. Math. France, Paris, **5** (1997), 115pp.