Representation categories of quantum groups

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EXERCISES

Exercise 1. Let $(A, \varphi, \eta, \Delta, \varepsilon)$ be a bialgebra and U, V, W be left A-modules. Check that the canonical isomorphisms of vector spaces $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l_V : k \otimes V \to V$ and $r_V : V \otimes k \to V$ are left A-module morphisms.

Exercise 2. a) Show that a universal R-matrix verifies $(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = 1_A$.

Hint: Apply $id \otimes \varepsilon \otimes id$ to the two last equalities of the definition of a universal R-matrix.

b) Show that a universal R-matrix verifies $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ - the **quantum Yang-Baxter equation**.

c) Let $(A, \Delta, \varepsilon, S)$ be a braided Hopf algebra with invertible antipode S and with a universal R-matrix R. Using Exercises 1 and 2, and relations $m(S \otimes id_A)\Delta(a) = m(id_A \otimes S)\Delta(a) = m(S^{-1} \otimes id_A)\Delta^{op}(a) = m(id_A \otimes S^{-1})\Delta^{op}(a) = \varepsilon(a)1$ (for all $a \in A$), show that $R^{-1} = (S \otimes id_A)(R) = (id_A \otimes S^{-1})(R)$.

Exercise 3. a) Let A be the algebra generated by two elements x and y and relations

$$x^2 = 1, \quad y^2 = 0, \quad yx + xy = 0.$$

The set $\{1, x, y, xy\}$ forms a basis of the vector space underlying A. Is this algebra semisimple?

b) Check that the formulas

$$\Delta(x) = x \otimes x, \ \Delta(y) = 1 \otimes y + y \otimes x, \ S(x) = x, \ S(y) = xy, \ \varepsilon(x) = 1, \ \varepsilon(y) = 0.$$

define a non commutative and non co-commutative Hopf algebra structure on A.

c) Show that S is of order 4 and that, for any $a \in A$, we have $S^2(a) = xax^{-1}$.

d) Let us put

$$R_q = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \frac{q}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y),$$

where $q \in k$. Show that R_q is a unversal R-matrix for A, so we have a family of concrete examples of braided Hopf algebras parameterized by q. Show that $R_q^{-1} = \tau_{A,A}(R_q)$.

Exercise 4. Let (A, Δ, ε) be a bialgebra acting on a unital algebra M and $M \rtimes A = M \otimes A$ be the vector space equipped with the product

$$[m \otimes a][n \otimes b] = [m(a_{(1)} \cdot n) \otimes a_{(2)}b],$$

Check that this product is associative with unit $1_M \otimes 1_A$.

Exercise 5. Check that the quantum double $D(G)$ of a finite group G is indeed a braided Hopf algebra and that $S^2 = id$.

Exercise 6. Let V be a right comodule over a finite dimensional bialgebra (A, Δ, ε). Show that V is automatically a left A^*-module with the left action $f \cdot v := (id \otimes f)(\Delta_V(v)) \ \forall f \in A^*$.

Exercise 7. Let us consider the strict tensor category $Vec_f(k)$, let V be an object of this category and V^* be its dual vector space. Let us define the maps $b_V : k \to V \otimes V^*$ and $d_V : V^* \otimes V \to k$ by

$$b_V(1) = \sum_i v_i \otimes v^i \quad \text{and} \quad d_V(v^i \otimes v_j) = < v^i, v_j >,$$

where $\{v_i\}_i$ is any basis of V and $\{v^i\}_i$ is the dual basis of V^*. Check that these definitions do not depend on the choice of the bases and that these maps define a left duality in $Vec_f(k)$.

Exercise 8. Let $\mathcal{C}$ be a strict tensor category with left duality. Check that $(id_V)^* = id_{V^*}$.

Exercise 9. Show that any symmetric tensor category $\mathcal{C}$ with left duality is a ribbon category with the trivial twist $\theta_V = id_V$. In particular, such is the category $Rep_f(A)$, where A is a cocommutative Hopf algebra or a braided Hopf algebra whose universal R-matrix r verifies $\tau_{A,A}(R) = R^{-1}$.

Exercise 10. For any object V of a ribbon category $\mathcal{C}$ and any endomorphism f of V, the **quantum trace** $tr_q(f)$ of f is defined as the following element of the monoid $End(I)$:

$$tr_q(f) = d_V c_{V,V^*}((\theta_V \circ f) \otimes id_{V^*})b_V.$$

Show that this definition gives the usual trace if $\mathcal{C} = Vec_f(k)$.

Exercise 11. Let us consider the braided Hopf algebra of Exercise 3. Show that the Drinfeld element u verifies $u = S(u) = x$ independently on q and that this Hopf algebra is ribbon with $\theta = 1$.