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Ideas That Will Outlast Us

To the Memory of George Kac (Georgiy Isaakovich Kac)

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Translated into English by Nataliya Markova

Preface

Here is a story about a person whose work and destiny were closely related to two important advances in 20th century mathematics: supermathematics and quantum groups. The first serious step towards supermathematics (the introduction of Lie superalgebras and formal Lie supergroups) was made by F.A. Berezin and G.I. Kac in [3]. Interesting memories have been recorded about Felix Alexandrovich Berezin but almost nothing has been written about Georgiy Isaakovich Kac (in fact, only [2]). Discussing the impact of [3] on mathematics and mathematical physics, Yu.A. Neretin wrote: "I have heard very little about the second author of this paper, G.I. Kac. It is worth noting that Kac's papers on 'ring groups' have to a great extent set the stage for another 'explosion', namely, the works on 'quantum groups'." See also V.G. Drinfeld's discussion in [4].

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I knew George Kac (a remarkable mathematician and a remarkable person) between 1968 and 1978. He died of a heart attack on 20 May 1978, in the prime of his talent and vitality. His friend B.I. Khatset wrote that his modesty and generosity had earned him the nickname Pierre Bezukhov (one of the principal characters of Leo Tolstoy's *War and Peace*) and, as you will see below, he was like this in mathematics, too. After G.K.'s death, whilst spending considerable time at research and academic centres where follow-up work was taking place, I observed the strong development of his ideas. That is why you will find here not just personal memories but also reflections on his mathematical ideas, their genesis, their evolution and their impact on other researchers. This is not a scientific paper so it does not claim to be rigorous or exhaustive. Nevertheless, it will be more understandable to those who are to a certain extent familiar with algebra and analysis. I am writing mainly about events that I have witnessed.

Kac algebras

In the autumn of 1968 I started attending G.K.'s seminar on operator rings. He was a remarkable speaker and his manner of setting out the material was clear and rigorous and, moreover, lively and accessible. Nothing I had heard before could match it and long after his death only a handful of speakers left a comparable impression.

We came to know each other more closely after my graduation from Kiev University, when I was hired by the Kiev Aviation Engineering Military College, where Kac served as a professor of mathematics. Even though I graduated from the university with honours, in those years, at the height of anti-Semitism, it was impossible for me to formally enrol in a PhD programme. For the same reason G. K. could not work either at the university or at the academy of sciences. Happily, Yu.M. Berezanski helped him set up his seminar at the institute of mathematics. Over the following years, I was fortunate to work with G.K., which not only guided my scientific career but also determined my future life path. I have to mention that at the same period I was doing my PhD thesis under the informal supervision of M.L. Gorbachuk, to whom I am deeply grateful.

G.K. offered me the choice of two research topics. The first was to further develop his joint work with Berezin [3], one of the pioneering works in the area now called supermathematics and dedicated to the study of mathematical structures (superalgebras, Lie superalgebras, supermanifolds, etc.) graded, in the simplest case, by the group Z/2Z. The motivation was provided by quantum mechanics, where two kinds of particles, fermions and bosons, are governed by totally different statistical laws. Both Kac and Berezin had a strong background in theoretical physics, so no wonder they became pioneers in this new area. Kac's PhD thesis, defended in 1950 (with N.N. Bogolyubov as an advisor), was dedicated to the correlation theory of electron gas. Let us also mention the superanalogue of Frobenius' theorem, obtained by G.K. and A.I. Koronkevich shortly after [3].

Nevertheless, I missed my chance to become a "supermathematician" and chose another topic, *ring groups*, introduced by G.K. around 1960. Below is a reminder of the sources of this theory.

Let G be a commutative, locally compact group and $\underline{G}$ its dual, i.e. the group of its unitary continuous characters (which is also commutative and locally compact). Pontryagin's duality says that the dual to $\underline{G}$ is isomorphic to G. However, this beautiful theory breaks down if G is non-commutative, even if it is finite, since the characters of such a group are too few and do not contain all the information about the group. To save the duality theory one could replace the characters with the irreducible unitary representations (in the commutative case they are just characters). Indeed, T. Tannaka showed in 1938 that a compact group can be restored up to isomorphism from a collection of its representations, while in 1949 M.G. Krein gave an axiomatic description of such a dual of a compact group. Nevertheless, the fact that this dual, a block-algebra, is not a group breaks the symmetry of the duality. Later on, such a non-symmetric duality theory was developed by W.F. Stinespring (1959) for unimodular groups and by N. Tatsuuma (1965–66) for general locally compact groups.

Translating Stinespring's paper into Russian for the collection "Matematika", G.K. came up with the idea to construct a new category of *ring groups* that would contain both groups and their duals with a symmetric duality acting within (i.e. an object and its dual must have the same mathematical structure, like in Potryagin's duality). A ring group is a collection (A, Γ, S, m), where A is an operator algebra, an *invariant measure* m is a *trace* on it (i.e. a positive central linear form which can be unbounded), a *coproduct* Γ is an algebra homomorphism from A to its tensor square, and an *antipode* S is an involutive anti-isomorphism of A, satisfying certain axioms.

When A is commutative, it can be realised as the algebra of essentially bounded measurable functions on a unimodular group G with respect to the invariant measure of G, $\Gamma f(x) = f(xy)$, $Sf(x) = f(x^{-1})$ (where x, y are elements of G) and m is the integral over the invariant measure on G. So, unimodular groups are included in the category of ring groups. Their duals are exactly cocommutative ring groups (i.e. Γ is stable under the permutation of factors in the tensor product). In this case, the algebra A is generated by the translation operators $L(x)$ or, equivalently, by the convolution operators $L(f)$ with f a continuous integrable function on G. Γ maps $L(x)$ to its tensor square, $S(L(x)) = L(x^{-1})$ and $m(L(f)) = f(e)$, where e is the unit of G.

In purely algebraic terms, ring groups are nothing other than *Hopf algebras* studied earlier in topology. G.K. was not aware of the existence of Hopf algebras and reinvented them when he introduced ring groups.

Lastly, G. K. gave a construction of a dual in this category. Applying this construction twice, one has an object isomorphic to the original one, like in Pontryagin's duality. These results were first announced in *Soviet Mathematical Doklady* in 1961 and later published in detail in [7]. That work used the techniques from I. Segal's paper on traces on operator algebras, also translated into Russian by G. K. for the "Matematika" collection.

The research topic I chose was the open problem formulated in G.K.'s habilitation thesis (Moscow University, 1963): to extend ring group theory in a way to cover all locally compact groups.

Fairly soon, I identified the technical tools to be prepared. Firstly, the above mentioned traces had to be replaced by possibly unbounded positive forms on operator algebras that are not necessarily central, called *weights*. Secondly, G.K. made systematic use of close ties between traces and so-called *Hilbert algebras*, so one had, passing from traces to weights, to develop an appropriate generalisation of Hilbert algebras. The idea appealed to G.K. and we started working on it with vigour. Shortly afterwards, the papers by M. Takesaki and F. Combes appeared, containing the needed techniques; they were clear to us since we had already gone halfway.

Now the coveted target was within reach but we had to hurry, since we were not alone in our pursuit. By that time, Takesaki had already written a paper on generalisation of ring groups; in addition, he had mastered all the necessary techniques. It remains a mystery to me why he did not come first in the race, being a leading specialist with a number of brilliant results to his credit. G.K. was also convinced that J. Dixmier saw the same objective. He said: "We must hurry. I'm sure that Dixmier has someone tackling this problem." As it turned out, he was absolutely right. But even in that stressful situation his integrity did not fail him: since he considered that it was me who had suggested the basic ideas, he decided to allow me the opportunity to complete the solution by myself, thus becoming the sole author, while it was he who had set the problem and had put much effort into adequately preparing me for solving it. He liked me to come to the lecture room at the end of his lectures to discuss and to walk with him from Uritski Square, where the Military College was situated, past the railway station to his home on Bolshaya Podvalnaya Street (the street names are of the period). Sometimes these discussions continued in his apartment.

Working on my own involved some risk because I had not yet fully mastered the necessary techniques and needed more time. To make matters worse, I was deeply upset by the death of my father in August 1971. Finally, seeing that I was not making much progress, G.K. understood that we could well lose the race and took over. He promptly got through a couple of issues that had baffled me and with his support I started to advance much quicker. In a joint effort, we had fairly rapidly completed a draft version. Even then G.K. remained true to himself; he suggested that I first publish part of the solution on my own, and only after that we publish the entire solution, which we did – the note [16] was submitted earlier than both papers [17].

Then we were in for an ordeal. Takesaki's paper [14] appeared, in which he took one more step towards the generalisation of ring groups. It was unavailable in Kiev, while the review on it in "Реферативный Журнал" (the Soviet analogue of *Mathematical Reviews*) indicated the construction of a duality theory that generalised the one of ring groups and covered all locally compact groups. Given Takesaki's reputation and the title of the paper, there was little doubt that we had lost. G.K., upset, dropped all his work and rushed to Moscow to read it firsthand (usually he visited Moscow several times a year in order to keep track of publications unavailable in Kiev and to socialise with colleagues, such as M.A. Naimark, F.A. Berezin, A.A. Kirillov and others). Back in Kiev with a photocopy of Takesaki's paper in hand, G.K. said that the reviewer was wrong and that the paper's results had not achieved the goal.

We completed and published our papers. But about the same time, papers came out by M. Enock and J.-M. Schwartz (who worked under Dixmier's supervision) containing equivalent results, although using a somewhat different technique. As Michel Enock later told me, Dixmier had also urged them to hurry, explaining that besides Takesaki, there had to be someone in Kiev working on the same problem with Kac. Our French colleagues used to send us their preprints and papers, while we could not respond to them, since we worked at a Military College and, bound by secrecy regulations, were not allowed to communicate with foreigners. For example, in 1975 we were invited to participate in a conference in Marseille dedicated to the subject of our studies and I rashly showed my invitation at the so-called First Department of the College. I was lucky to get away with it; the KGB men told me that if I wanted to continue my work there, I must throw the invitation away.

In view of the fundamental role of G.K. in the discovery of the new mathematical objects, the suggestion of Enock and Schwartz to name them *Kac algebras* was highly appropriate. I learned about it from G.K. himself. He had very expressive eyes and it was apparent how much he enjoyed the news. Of course, being a man of great modesty, he never actually pronounced this name. Today, an entire book [5] is dedicated to this subject.

Kac algebras were invented not just for the sake of beauty but with a view to applying them to the solution of various problems. According to G.K., ring groups had to be regarded in the same way as the ordinary groups that they generalise, while their application areas might be wider, which later proved to be the case. Nevertheless, bringing ring groups into effective use required a deeper understanding of their structure, examples and properties. Back in the early 1960s, G.K. had distinguished and started to explore special classes of ring groups: *compact, discrete and finite*.

G.K.'s habilitation thesis contained a list of open problems – some of them have been solved since (like the one examined above). One of these open problems is the fact that the existence of an invariant measure is an axiom of a Kac algebra and is not derived from other axioms, as in the case of ordinary groups (the uniqueness of this measure is easy to prove). This is justified by Weil's and Haar's theorems, which state that the existence of an invariant measure is equivalent to the existence of a locally compact topology on a group. However, it would be natural to define Kac algebras only in algebraic and topological terms and *to prove* the existence of an invariant measure, thereby generalising the Haar theorem for ordinary groups. For the above special classes of Kac algebras this was done in [11, 13].

In particular, finite Kac algebras [11] are finite-dimensional semisimple Hopf *-algebras over the field of complex numbers. Their axiomatics contains a *counit* that is the analogue of the unit in an ordinary group. The existence of an invariant measure is a theorem.

In his habilitation thesis and in [9], G. K. extended a number of classical results on finite groups to finite Kac algebras. In particular, he obtained the analogue of the Lagrange theorem stating that the order of a subgroup divides the order of a group. Later on, a stronger statement was established by V.D. Nichols and M.B. Zoeller for any finite-dimensional Hopf algebra. As for finite groups, the only Kac algebra of prime dimension is the cyclic group. This result has been generalised many times in recent works by various authors. G.K. also proved that for any irreducible representation of a finite Kac algebra, there is a basis in which its matrix elements are algebraic integers.

We saw above that commutative and co-commutative Kac algebras correspond to ordinary groups and their duals respectively. Of special interest are examples of *nontrivial* Kac algebras that do not belong to these two classes. The first such examples were built by G.I. Kac and V.G. Paljutkin [8, 10, 11]. As noted by V.G. Drinfeld [4], they were the first known examples of *quantum groups*.

Ordinary groups can be realised as set transformation groups and Kac algebras can also act, but on algebras instead of sets. G.K. repeatedly voiced the opinion, which is now generally accepted, that algebras are noncommutative counterparts of sets and ring groups are noncommutative counterparts of groups. He suggested a definition of *a ring group action* on an algebra and a construction of the crossed product of these objects. Later on, Enock and Schwartz dedicated a series of papers to that type of construction and related results.

Discussing the choice of new problems in the mid-1970s, G.K. strongly advocated an in-depth study of finite ring groups and their representations in the vein of his works. Another option was the search for new examples of nontrivial Kac algebras but that turned out to be an uphill task, and only 20 years later did I understand how to systematically construct them. However, in those days analytical aspects of the theory seemed closer to me than algebraic ones.

I decided to explore such generalisations of Kac algebras, which would cover so-called *generalised shift operators* or *hypergroups*, studied earlier by J. Delsarte, Yu.M. Berezanski, S.G. Krein and B.M. Levitan. This line of research offered challenging analytical problems and interesting applications. But, despite the fact that this problem was mentioned in his habilitation thesis and in [8], G.K. thought that such a theory would be somewhat deficient because the coproduct must be just a positive map and not a homomorphism of operator algebras like in the Kac algebra theory, so that certain essential properties of Kac algebras would be lost.

Still, the fact that due to weaker constraints the theory became far richer in applications seemed of importance to me. In the 1980s and 1990s, encouraged by Yu.M. Berezanski, I worked in this direction. But in the mid-1970s, to my great sorrow, our active collaboration with G.K. was practically over, even though we continued to see each other and discuss mathematical and non-mathematical topics. Since then, I had always kept track of Kac algebra theory, my first love in mathematics, but it was not until the mid-1990s that I returned to it.

Quantum Groups

Let us now turn to the evolution of G.K.'s ideas after his death. Kac algebras solved the problem of duality but their range of applications was not wide enough (which was the reason for me to take up a generalisation). However, the genuine breakthrough in understanding the approach to extend this circle of ideas occurred in the mid-1980s, when V.G. Drinfeld and others discovered quantum groups [4]. In purely algebraic terms, the squared antipode in a quantum group is not necessarily trivial (it is trivial in a Kac algebra) – a seemingly small matter but one that makes the fundamental difference. Many important examples and applications appeared, in particular in theoretical physics and topology. As Alain Connes put it in the preface to [5], Kac algebras have proved "not sufficiently non-unimodular" to cover new applications and therefore the necessity arose for quantum groups. Note that Kac algebras are "non-unimodular" generalisations of ring groups in their own right!

As for quantum group theories in operator algebraic framework, S.L. Woronowicz [18] built the theory of compact quantum groups that contained the theorem of existence of invariant measure (for compact Kac algebras this theorem had been proved earlier [13]) and studied their irreducible representations. Constructing concrete examples of quantum groups, he overcame numerous functional analytical difficulties which highlighted the challenges in extending Kac algebra theory to capture all interesting examples while keeping its beauty and symmetry.

In the 1950s, Stinespring noted the essential role played in the duality of unimodular groups by the unitary sending $f(x,y)$ to $f(x,xy)$. G.K. had built an analogue of this operator for arbitrary ring groups and discovered its fundamental property, the so-called *pentagonal relation*. Later on, G.K. and I, as well as Enock and Schwartz, made full use of this observation passing to non-unimodular Kac algebras. But S. Baaj and G. Skandalis [1] took it even further, pointing out that a *multiplicative unitary* (i.e. satisfying the pentagonal relation) under certain regularity conditions already allows one to build two operator algebras in duality, each of them carrying a more general structure than that of a Kac algebra.

After the end of the Soviet era, Michel Enock and I exchanged all our past publications. He could not know that I had already received everything he had sent me before, since not a single response came back from behind the Iron Curtain (the envelopes I often received had traces of being open and re-sealed, certainly by the KGB, but they contained only mathematical articles). We started to correspond and soon Michel came to Kiev. One of the sentimental events of his stay was our visit to G.K.'s tomb at Gostomelskoye cemetery.

When I came to Paris in the spring of 1992, many colleagues said that they were happy to see a representative of George Kac's school. They did not know that G.K. could never have formal graduate students, and his "school" eventually consisted of just V.G. Paljutkin, me, A.I. Koronkevich, and V. Zhuk. It was so unfair that G.K. had not enjoyed even a small part of the recognition that he so much deserved!

Besides the above mentioned difficulties of his life, his scientific results had not been properly appreciated by some leading specialists, e.g. by L.S. Pontryagin, whose duality principle had been so brilliantly extended by G.K., and by I.M. Gel'fand, who had been rather critical of G.K.'s works. In 1983, Gel'fand tried to persuade me to take up a different subject, saying: "Do not bury your talent as your teacher Kac did!" This was just before the discovery of quantum groups, for which V.G. Drinfeld received the Fields Medal and which had Kac algebra theory as a direct predecessor (see [4])!

Gradually the idea arose to get back to constructing non-trivial examples of Kac algebras and quantum groups. When Woronowicz and others built examples of quantum groups, each case was unique and presented specific difficulties. It was desirable to have constructions generating a number of different examples in a unified way. For example, Drinfeld proposed a purely algebraic way to deform the coproduct and antipode without changing the algebra of a given quantum group in order to get a new one. Applying this construction, called *twisting*, to the dual of an ordinary group one could hope to come up with interesting examples of quantum groups. I started to work on analytical aspects of twisting.

Enock came to Kiev once again in May 1994 on the occasion of G.K.'s 70th anniversary. At the meeting of the Kiev Mathematical Society, Enock and I gave presentations about different facets of G.K.'s activities and, in particular, discussed the problem of examples of Kac algebras. Afterwards, we continued our discussions and in the spring of 1995 the paper [6] was finished. Later on, I extended and reinforced these results. In particular, we constructed a series of new "quantisations" of the Heisenberg group, which is popular with physicists. All of these objects turned out to be not only Kac algebras but even unimodular ring groups in the sense of G.K.'s very first definition.

The finite-dimensional aspect of twisting was discussed in Kiev with D. Nikshych, who built deformations of classical series of finite groups – symmetric, dihedral, quasiquaternionic and alternated. The latter gave the first known examples of *simple* Kac algebras (i.e. without proper normal Kac subalgebras).

In the late 1990s, there was progress in the long-awaited generalisation of Kac algebra theory. Baaj and Skandalis, on the one hand, and Woronowicz, on the other hand, better understood the conditions to be placed on a multiplicative unitary so that it generates a pair of quantum groups in duality. A version of such a theory was proposed by T. Masuda, I. Nakagami and S.L. Woronowicz. In Leuven (Belgium), A. Van Daele discussed at his seminar new examples of quantum groups and approaches to the construction of a general theory. Among other things, he distinguished a class of quantum groups defined in a purely algebraic way such that their topological properties could be derived.

Finally, a satisfactory theory of locally compact quantum groups was proposed in 1999 by J. Kustermans and S. Vaes [17]. It was as beautiful and symmetric as Kac algebra theory was; without much exaggeration one can say that it had been modelled on the latter. A locally compact quantum group is a collection $(A, \Gamma, m, n,)$, where A, Γ, m are the same as in Kac algebra theory, and n is a right invariant weight on A. The axioms do not mention antipode explicitly but imply its existence and properties and not one but two weights are present – here it is: the second non-unimodularity! It resembles an ordinary locally compact group with two invariant measures: left and right.

In 1999–2000, I spent a few months in Leuven and could familiarise myself with these things firsthand, whereupon I got an idea to generalise the *theory of extensions* [8], which G.K. used for getting non-trivial examples of Kac algebras [10, 11]. Given a commutative Kac algebra K_1, which is an algebra of functions on an ordinary group G_1, and a co-commutative K_2, which is a dual of an ordinary group G_2, the question is whether it is possible to build an extension of K_1 by K_2 – a new Kac algebra such that K_1 would be its normal Kac subalgebra and K_2 would be the corresponding quotient. G.K. showed that such an extension exists if and only if the groups G_1 and G_2 act on each other as on sets and these actions are compatible in a special way. He had described all these extensions in terms of what is referred to as *bicrossed product construction*. Later on, this construction was rediscovered by M. Takeuchi and S. Majid.

Now that we had a far wider category of locally compact quantum groups in hand, one might expect to make the most of the construction. I shared these ideas with Stefaan Vaes and in a few months the paper [15] was finished. Taking various G_1 and G_2, one could get plenty of concrete examples of Kac algebras and quantum groups. In our later work, we classified extensions with G_1 and G_2 low-dimensional Lie groups; another quantum group with surprising regularity properties was built by Baaj, Skandalis and Vaes with G_1 and G_2 coming from number theory; in yet another case, my former PhD student Pierre Fima used similar groups to construct examples of quantum groups with prescribed types of their von Neumann algebras.

Baaj, Skandalis and Vaes also extended to locally compact quantum groups another brilliant idea of G.K., who had built in [8] what was later (when the importance of this sequence in various problems became evident) called the Kac exact sequence.

I will conclude my notes with a story about Kac algebras "in action". In the postscript to [5], A. Ocneanu explained that Kac algebras must arise as non-commutative analogues of group symmetries in the subfactor theory founded by V. Jones. Indeed, in 1994, W. Szymanski and R. Longo independently of one another proved that if N is a finite index subfactor of a factor M then under certain conditions there is necessarily a Kac algebra K acting on M such that N is a subalgebra of fixed points with respect to this action. Vice versa, given a Kac algebra and its action on a factor, one can build a finite index subfactor. This is a far-reaching extension of classical Galois theory, where N and M are fields, and K is a Galois group. Later on, M. Enock and R. Nest came up with a similar result for subfactors of infinite index, in which case a Kac algebra had to be replaced by a locally compact quantum group.

Postscript

The purpose of these notes was to show, apart from personal memories, the powerful influence of Georgiy Isaakovich's works on the progress of a wide area of mathematics. He has been gone for more than 35 years but in scores of recently published works, you will effortlessly find clear evidence of their foundation in his ideas. Indeed, as Pushkin put it: "I have built a monument not wrought by hands…"

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